

OPERADS, ALGEBRAS AND MODULES IN GENERAL MODEL CATEGORIES

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ABSTRACT. In this paper we develop the theory of operads, algebras and modules in cofibrantly generated symmetric monoidal model categories. We give J -semi model structures, which are a slightly weaker version of model structures, for operads and algebras and model structures for modules. We prove homotopy invariance properties for the categories of algebras and modules. In a second part we develop the theory of \mathbb{S} -modules and algebras of [EKMM] and [KM], which allows a general homotopy theory for commutative algebras and pseudo unital symmetric monoidal categories of modules over them. Finally we prove a base change and projection formula.

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1. INTRODUCTION

Recently important new applications of model categories appeared, the most notable one maybe in the work of Voevodsky and others on the \mathbb{A}^1 -local stable homotopy category of schemes. But also for certain questions in homological algebra

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model categories are quite useful, for example when one deals with unbounded complexes in abelian categories. In topology, mainly in the stable homotopy category, one is used to deal with objects having additional structures, for example modules over ring spectra. The work of [EKMM] made it possible to handle commutativity appropriately, namely the special properties of the linear isometries operad lead to a strictly associative and commutative tensor product for modules over E_∞ -ring spectra. As a consequence many constructions in topology became more elegant or even possible at all (see [EKM]). Moreover the category of E_∞ -algebras could be examined with homotopical methods because this category carries a model structure. In [KM] a parallel theory in algebra was developed (see [May]).

Parallel to the achievements in topology the abstract model category theory was further developed (see [Hov1] for a good introduction to model categories, see also [DHK]). Categories of algebras and of modules over algebras in monoidal model categories have been considered ([SS], [Hov2]). Also localization techniques for model categories have become important, because they yield many new useful model structures (for example the categories of spectra of [Hov3]). The most general statement for the existence of localizations is given in [Hir].

In all these situations it is as in topology desirable to be able to work in the commutative world, i.e. with commutative algebras and modules over them. Since a reasonable model structure for commutative algebras in a given symmetric monoidal model category is quite unlikely to exist the need for a theory of E_∞ -algebras arises. Also for the category of modules over an E_∞ -algebra a symmetric monoidal structure is important. One of the aims of this paper is to give adequate answers to these requirements.

E_∞ -algebras are algebras over particular operads. Many other interesting operads appeared in various areas of mathematics, starting from the early application for recognition principles of iterated loop spaces (which was the reason to introduce operads), later for example to handle homotopy Lie algebras which are necessary for general deformation theory, the operads appearing in two dimensional conformal quantum field theory or the operad of moduli spaces of stable curves in algebraic geometry. In many cases the necessary operads are only well defined up to quasi isomorphism or another sort of weak equivalence (as for example is the case for E_∞ -algebras), therefore a good homotopy theory of operads is desirable. A related question is then the invariance (up to homotopy) of the categories of algebras over weakly equivalent operads and also of modules over weakly equivalent algebras. We will also give adequate solutions to these questions. This part of the paper was motivated by and owes many ideas from [Hin1] and [Hin2].

So in the first part we will develop the theory of operads, algebras and modules in the general situation of a cofibrantly generated symmetric monoidal model category satisfying some technical conditions which are usually fulfilled. Our first aim is to provide these categories with model structures. It turns out that in general we cannot quite get model structures in the case of operads and algebras, but a slightly weaker structure which we call a J -semi model structure. A version of this structure already appeared in [Hov2]. To the knowledge of the author no restrictions arise in the applications when using J -semi model structures instead of model structures. The J -semi model structures are necessary since the free operad and algebra functors are not linear (even not polynomially). These structures appear in two versions, an absolute one and a version relative to a base category.

We have two possible conditions for an operad or an algebra to give model structures on the associated categories of algebras or modules, the first one is being cofibrant (which is in some sense the best condition), and the second one being cofibrant in an underlying model category.

In the second part of the paper we demonstrate that the theory of \mathbb{S} -modules of [EKMM] and [KM] can also be developed in our context if the given symmetric monoidal model category \mathcal{C} either receives a symmetric monoidal left Quillen functor from \mathbf{SSet} (i.e. is simplicial) or from $Comp_{\geq 0}(\mathbf{Ab})$. The linear isometries operad \mathcal{L} gives via one of these functors an E_∞ -operad in \mathcal{C} with the same special properties responsible for the good behavior of the theories of [EKMM] and [KM]. These theories do not yield honest units for the symmetric monoidal category of modules over \mathcal{L} -algebras, and we have to deal with the same problem. In the topological theory of [EKMM] it is possible to get rid of this problem, in the algebraic or simplicial one it is not. Nevertheless it turns out that the properties the unit satisfies are good enough to deal with operads, algebras and modules in the category of modules over a cofibrant \mathcal{L} -algebra. This seems to be a little contraproductive, but we need this to prove quite strong results on the behavior of algebras and modules with respect to base change and projection morphisms. These results are even new for the cases treated in [EKMM] and [KM].

In an appendix we show that one can always define a product on the homotopy category of modules over an \mathcal{O} -algebra for an arbitrary E_∞ -operad \mathcal{O} without relying on the special properties of the linear isometries operad, but we do not construct associativity and commutativity isomorphisms in this situation! In the case when \mathbb{S} -modules are available this product structure is naturally isomorphic to the one defined using \mathbb{S} -modules.

Our constructions have explicit applications, for example for the \mathbb{A}^1 -homotopy categories of Voevodsky, for triangulated categories of motives over a general base, for the “tangential base point” constructions à la Grothendieck, Deligne and others and its “motivic” interpretation, which we demonstrate in a forthcoming paper, to develop the theory of schemes in symmetric monoidal cofibrantly generated model categories (see [TV]), etc.

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2. PRELIMINARIES

We first review some standard arguments from model category theory which we will use throughout the paper (see for the first part e.g. the introduction to [Hov2]).

Let \mathcal{C} be a cocomplete category. For a pushout diagram in \mathcal{C}

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow \varphi & & \uparrow \\ K & \xrightarrow{g} & L \end{array}$$

we call f the pushout of g by φ , and we call B the pushout of A by g with attaching map φ . If we say that B is a pushout of A by g and g is an object of \mathcal{C} then we mean that $B = g$ and A need not be defined in this case (the sense of this statement will

become clear in the statements describing pushouts of operads and algebras over operads in a model category \mathcal{C}).

Let I a set of maps in \mathcal{C} . Let $I\text{-inj}$ denote the class of maps in \mathcal{C} which have the right lifting property with respect to I , $I\text{-cof}$ the class of maps in \mathcal{C} which have the left lifting property with respect to $I\text{-inj}$ and $I\text{-cell}$ the class of maps which are transfinite compositions of pushouts of maps from I . Note that $I\text{-cell} \subset I\text{-cof}$ and that $I\text{-inj}$ and $I\text{-cof}$ are closed under retracts.

Let us suppose now that the domains of the maps in I are small relative to $I\text{-cell}$. Then by the small object argument there exists a functorial factorization of every map in \mathcal{C} into a map from $I\text{-cell}$ followed by a map from $I\text{-inj}$. Moreover every map in $I\text{-cof}$ is a retract of a map in $I\text{-cell}$ such that the retract induces an isomorphism on the domains of the two maps. Also the domains of the maps in I are small relative to $I\text{-cof}$.

Now let \mathcal{C} be equipped with a symmetric monoidal structure such that the product $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ preserves colimits (e.g. if the monoidal structure is closed). We denote the pushout product of maps $f : A \rightarrow B$ and $g : C \rightarrow D$,

$$A \otimes D \sqcup_{A \otimes C} B \otimes C \rightarrow B \otimes D ,$$

by $f \square g$.

For ordinals ν and λ we use the convention that the well-ordering on the product ordinal $\nu \times \lambda$ is such that the elements in ν have higher significance. We will need the

Lemma 1. *Let $f : K_0 \rightarrow K_\mu = \text{colim}_{i < \mu} K_i$ and $g : L_0 \rightarrow L_\lambda = \text{colim}_{i < \lambda} L_i$ be transfinite compositions with transition maps $f_i : K_i \rightarrow K_{i+1}$ and $g_i : L_i \rightarrow L_{i+1}$. Then the pushout product $f \square g$ is a transfinite composition $M_0 \rightarrow M_{\mu \times \lambda} = \text{colim}_{i < \mu \times \lambda} M_i$ over the product ordinal $\mu \times \lambda$ where the transition maps $M_{(i,j)} \rightarrow M_{(i,j+1)}$ are pushouts by the maps $f_i \square g_j$.*

Proof. For any $(i, j) \leq \mu \times \lambda$ define $M_{(i,j)}$ to be the colimit of the diagram

$$\begin{array}{ccccc} A_\mu \otimes B_0 & & A_{i+1} \otimes B_j & & A_i \otimes B_\lambda \\ & \nwarrow & \nearrow & \nwarrow & \nearrow \\ & A_{i+1} \otimes B_0 & & A_i \otimes B_j & \end{array}$$

Clearly $M_{(0,0)} = A_\mu \otimes B_0 \sqcup_{A_0 \otimes B_0} A_0 \otimes B_\lambda$ is the domain and $M_{\mu \times \lambda} = A_\mu \otimes B_\lambda$ the codomain of $f \square g$. Moreover it is easy to see that the pushout of $M_{(i,j)}$ by $f_i \square g_j$ with the obvious attaching map is canonically isomorphic to $M_{(i,j+1)}$. Since \otimes preserves colimits the assignment $(i, j) \mapsto M_{(i,j)}$ is a transfinite composition. \square

The pushout product is associative. For maps $f_i : A_i \rightarrow B_i$, $i = 1, \dots, n$, in \mathcal{C} giving a map from the domain of $g := \square_{i=1}^n f_i$ to an object $X \in \mathcal{C}$ is the same as to give maps φ_j from the

$$S_j := \left(\bigotimes_{i=1}^{j-1} B_i \right) \otimes A_j \otimes \left(\bigotimes_{i=j+1}^n B_i \right)$$

to X for $j = 1, \dots, n$ such that φ_j and $\varphi_{j'}$ ($j' > j$) coincide on

$$I_{j,j'} := \left(\bigotimes_{i=1}^{j-1} B_i \right) \otimes A_j \otimes \left(\bigotimes_{i=j+1}^{j'-1} B_i \right) \otimes A_{j'} \otimes \left(\bigotimes_{i=j'+1}^n B_i \right)$$

after the obvious compositions. We call the S_j the *summands* of the domain of g and the $I_{j,j'}$ the *intersections* of these summands. Sometimes some of the f_i will coincide. Then there is an action of a product of symmetric groups on g , and the quotient of a summand with respect to the induced action of the stabilizer of this summand will also be called a summand (and similarly for the intersections).

For the rest of the paper we fix a cofibrantly generated symmetric monoidal model category \mathcal{C} with generating cofibrations I and generating trivial cofibrations J . For simplicity we assume that the domains of I and J are small relative to the whole category \mathcal{C} . The interested reader may weaken this hypothesis appropriately in the statements below.

For a monad \mathbb{T} in \mathcal{C} we write $\mathcal{C}[\mathbb{T}]$ for the category of \mathbb{T} -algebras in \mathcal{C} . The following theorem summarizes the general method to equip categories of objects in \mathcal{C} with “additional structure” with model structures (e.g. as in [Hov2][Theorem 2.1]).

Theorem 1. *Let \mathbb{T} be a monad in \mathcal{C} , assume that $\mathcal{C}[\mathbb{T}]$ has coequalizers and suppose that every map in $\mathbb{T}J$ -cell, where the cell complex is built in $\mathcal{C}[\mathbb{T}]$, is a weak equivalence in \mathcal{C} . Then there is a cofibrantly generated model structure on $\mathcal{C}[\mathbb{T}]$, where a map is a weak equivalence or fibration if and only if it is a weak equivalence or fibration in \mathcal{C} .*

Proof. We apply [Hov1][Theorem 2.1.19] with generating cofibrations $\mathbb{T}I$, generating trivial cofibrations $\mathbb{T}J$ and weak equivalences the maps which are weak equivalences in \mathcal{C} .

By [McL][VI.2, Ex 2], $\mathcal{C}[\mathbb{T}]$ is complete and by [BW][9.3 Theorem 2] cocomplete. Property 1 of [Hov1][Theorem 2.1.19] is clear, properties 2 and 3 follow by adjunction from our smallness assumptions on the domains of I and J . Since each element of J is in I -cof, hence a retract of a map in I -cell, each element of $\mathbb{T}J$ is in $\mathbb{T}I$ -cof, hence together with our assumption we see that property 4 is fulfilled. By adjunction $\mathbb{T}I$ -inj (resp. $\mathbb{T}J$ -inj) is the class of maps in $\mathcal{C}[\mathbb{T}]$ which are trivial fibrations (resp. fibrations) in \mathcal{C} . Hence property 5 and the second alternative of 6 are fulfilled. \square

In most of the cases we are interested in the hypothesis of this theorem that every map in $\mathbb{T}J$ -cell is a weak equivalence won't be fulfilled. The reason is that we are considering monads which are not linear. The method to circumvent this problem was found by Hovey in [Hov2][Theorem 3.3]. He considers categories which are not quite model categories. We will call them semi model categories.

Definition 1. *(I) A J -semi model category over \mathcal{C} is a left adjunction $F : \mathcal{C} \rightarrow \mathcal{D}$ and subcategories of weak equivalences, fibrations and cofibrations in \mathcal{D} such that the following axioms are fulfilled:*

1. *The adjoint of F preserves fibrations and trivial fibrations.*
2. *\mathcal{D} is bicomplete and the two out of three and retract axioms hold in \mathcal{D} .*

3. *Cofibrations in \mathcal{D} have the left lifting property with respect to trivial fibrations, and trivial cofibrations whose domain becomes cofibrant in \mathcal{C} have the left lifting property with respect to fibrations.*
4. *Every map in \mathcal{D} can be functorially factored into a cofibration followed by a trivial fibration, and every map in \mathcal{D} whose domain becomes cofibrant in \mathcal{C} can be functorially factored into a trivial cofibration followed by a fibration.*
5. *Cofibrations in \mathcal{D} whose domain becomes cofibrant in \mathcal{C} become cofibrations in \mathcal{C} , and the initial object in \mathcal{D} is mapped to a cofibrant object in \mathcal{C} .*
6. *Fibrations and trivial fibrations are closed under pullback.*

We say that \mathcal{D} is cofibrantly generated if there are sets of morphisms I and J in \mathcal{D} such that $I\text{-inj}$ is the class of trivial fibrations and $J\text{-inj}$ the class of fibrations in \mathcal{D} and if the domains of I are small relative to $I\text{-cell}$ and the domains of J are small relative to maps from $J\text{-cell}$ whose domain becomes cofibrant in \mathcal{C} .

\mathcal{D} is called left proper (relative to \mathcal{C}) if pushouts by cofibrations preserve weak equivalences whose domain and codomain become cofibrant in \mathcal{C} (hence all objects which appear become cofibrant in \mathcal{C}). \mathcal{D} is called right proper if pullbacks by fibrations preserve weak equivalences.

(II) A category \mathcal{D} is called a J -semi model category if conditions (2) to (6) of Definition 1 are fulfilled where the condition of becoming cofibrant in \mathcal{C} is replaced by the condition of being cofibrant.

The same is valid for the definition of being cofibrantly generated and of being right proper.

(Note that the only reasonable property to require in a definition for a J -semi model category to be left proper, namely that weak equivalences between cofibrant objects are preserved by pushouts by cofibrations, is automatically fulfilled as is explained below when we consider homotopy pushouts.)

Alternative: One can weaken the definition of a J -semi model category (resp. of a J -semi model category over \mathcal{C}) slightly by only requiring that a factorization of a map in \mathcal{D} into a cofibration followed by a trivial fibration should exist if the domain of this map is cofibrant (resp. becomes cofibrant in \mathcal{D}). We then include into the definition of cofibrant generation that the cofibrations are all of $I\text{-cof}$. Using this definition all statements from section 3 on remain true if one does not impose any further smallness assumptions on the domains of I and J . This follows in each of the cases from the fact that the domains of I and J are small relative to $I\text{-cof}$.

Of course a J -semi model category over \mathcal{C} is a J -semi model category. There is also the notion of an I -semi (and also (I, J) -semi) model category (over \mathcal{C}), where the parts of properties 3 and 4 concerning cofibrations are restricted to maps whose domain is cofibrant (becomes cofibrant in \mathcal{C}).

We summarize the main properties of a J -semi model category \mathcal{D} (relative to \mathcal{C}) (compare also [Hov2][p. 14]):

By the factorization property and the retract argument it follows that a map is a cofibration if and only if it has the left lifting property with respect to the trivial fibrations. Similarly a map is a trivial fibration if and only if it has the right lifting

property with respect to the cofibrations. These two statements remain true under the alternative definition if \mathcal{D} is cofibrantly generated.

A map in \mathcal{D} whose domain is cofibrant (becomes cofibrant in \mathcal{C}) is a trivial cofibration if and only if it has the left lifting property with respect to the fibrations, and a map whose domain is cofibrant (becomes cofibrant in \mathcal{C}) is a fibration if and only if it has the right lifting property with respect to the trivial cofibrations.

Pushouts preserve cofibrations (also under the alternative definition if \mathcal{D} is cofibrantly generated). Trivial cofibrations with cofibrant domain (whose domain becomes cofibrant in \mathcal{C}) are preserved under pushouts by maps with cofibrant codomain (whose codomain becomes cofibrant in \mathcal{C}).

In the relative case the functor F preserves cofibrations (also in the alternative definition if \mathcal{D} is cofibrantly generated), and trivial cofibrations with cofibrant domain.

Ken Brown's Lemma ([Hov1][lemma 1.1.12]) remains true, and its dual version has to be modified to the following statement: Let \mathcal{D} be a J -semi model category (over \mathcal{C}) and \mathcal{D}' be a category with a subcategory of weak equivalences which satisfies the two out of three property. Suppose $F : \mathcal{D} \rightarrow \mathcal{D}'$ is a functor which takes trivial fibrations between fibrant objects with cofibrant domain (whose domain becomes cofibrant in \mathcal{C}) to weak equivalences. Then F takes all weak equivalences between fibrant objects with cofibrant domain (whose domain becomes cofibrant in \mathcal{C}) to weak equivalences.

We define cylinder and path objects and the various versions of homotopy as in [Hov1][Definition 1.2.4]. Cylinder and path objects exist for cofibrant objects (for objects which become cofibrant in \mathcal{C}).

We give the J -semi version of [Hov1][Proposition 1.2.5]:

Proposition 1. *Let \mathcal{D} be a J -semi model category (over \mathcal{C}) and let $f, g : B \rightarrow X$ be two maps in \mathcal{D} .*

1. *If $f \stackrel{l}{\sim} g$ and $h : X \rightarrow Y$, then $hf \stackrel{l}{\sim} hg$. Dually, if $f \stackrel{r}{\sim} g$ and $H : A \rightarrow B$, then $fh \stackrel{r}{\sim} gh$.*
2. *Let $h : A \rightarrow B$ and suppose A and B are cofibrant (become cofibrant in \mathcal{C}) and X is fibrant. Then $f \stackrel{l}{\sim} g$ implies $fh \stackrel{l}{\sim} gh$. Dually, let $h : X \rightarrow Y$. Suppose X and Y are cofibrant (become cofibrant in \mathcal{C}) and B is cofibrant. Then $f \stackrel{r}{\sim} g$ implies $hf \stackrel{r}{\sim} hg$.*
3. *If B is cofibrant, then left homotopy is an equivalence relation on $\text{Hom}(B, X)$.*
4. *If B is cofibrant and X is cofibrant (becomes cofibrant in \mathcal{C}), then $f \stackrel{l}{\sim} g$ implies $f \stackrel{r}{\sim} g$. Dually, if X is fibrant and B is cofibrant (becomes cofibrant in \mathcal{C}), then $f \stackrel{r}{\sim} g$ implies $f \stackrel{l}{\sim} g$.*
5. *If B is cofibrant and $h : X \rightarrow Y$ is a trivial fibration or weak equivalence between fibrant objects with X cofibrant (such that X becomes cofibrant in \mathcal{C}), then h induces an isomorphism*

$$\text{Hom}(B, X) / \stackrel{l}{\sim} \xrightarrow{\cong} \text{Hom}(B, Y) / \stackrel{l}{\sim}.$$

Dually, suppose X is fibrant and cofibrant (becomes cofibrant in \mathcal{C}) and $h : A \rightarrow B$ is a trivial cofibration with A cofibrant (such that A becomes cofibrant in \mathcal{C}) or a weak equivalence between cofibrant objects, then h induces an

isomorphism

$$\mathrm{Hom}(B, X) / \sim \xrightarrow{\cong} \mathrm{Hom}(A, X) / \sim .$$

This Proposition is also true for the alternative definition of a J -semi model category (over \mathcal{C}). We changed the order between 4 and 5, because it is a priori not clear that right homotopy is an equivalence relation (under suitable condition), this follows only after comparison with the left homotopy relation.

As in [Hov1][Corollary 1.2.6 and 1.2.7] it follows that if B is cofibrant and X is fibrant and cofibrant (becomes cofibrant in \mathcal{C}), then left and right homotopy coincide and are equivalence relations on $\mathrm{Hom}(B, X)$ and the homotopy relation on \mathcal{D}_{cf} is an equivalence relation and compatible with composition. The statement of [Hov1][Proposition 1.2.8] that a map in \mathcal{D}_{cf} is a weak equivalence if and only if it is a homotopy equivalence is proved exactly in the same way. The same holds for the fact that $\mathrm{Ho} \mathcal{D}_{cf}$ is naturally isomorphic to \mathcal{D}_{cf} / \sim ([Hov1][Corollary 1.2.9]). Finally the existence of the cofibrant and fibrant replacement functor RQ implies that the map $\mathrm{Ho} \mathcal{D}_{cf} \rightarrow \mathrm{Ho} \mathcal{D}$ is an equivalence.

Definition 2. *A functor $L : \mathcal{D} \rightarrow \mathcal{D}'$ between J -semi model categories is a left Quillen functor if it has a right adjoint and if the right adjoint preserves fibrations and trivial fibrations.*

Of course in the relative situation F is a left Quillen functor. We show that a left Quillen functor induces an adjunction between the homotopy categories (also when we use the alternative definition). L preserves (trivial) cofibrations between cofibrant objects, hence by Ken Brown's Lemma it preserves weak equivalences between cofibrant objects. This induces a functor $\mathrm{Ho} \mathcal{D} \rightarrow \mathrm{Ho} \mathcal{D}'$. By the dual version of Ken Brown's Lemma the adjoint of L preserves weak equivalences between fibrant and cofibrant objects which gives a functor $\mathrm{Ho} \mathcal{D}' \rightarrow \mathrm{Ho} \mathcal{D}$. One easily checks that L preserves cylinder objects on cofibrant objects and that the adjoint of L preserves path objects on fibrant objects. As in Lemma [Hov1][Lemma 1.3.10] it follows that on the derived functors between $\mathrm{Ho} \mathcal{D}$ and $\mathrm{Ho} \mathcal{D}'$ there is induced a natural derived adjunction.

Next we are going to consider Reedy model structures and homotopy function complexes. We have the analogue of [Hov1][Theorem 5.1.3]:

Proposition 2. *Let \mathcal{D} be a J -semi model category and \mathcal{B} be a direct category. Then the diagram category $\mathcal{D}^{\mathcal{B}}$ is a J -semi model category with objectwise weak equivalences and fibrations and where a map $A \rightarrow B$ is a cofibration if and only if the maps $A_i \sqcup_{L_i A} L_i B \rightarrow B_i$ are cofibrations for all $i \in \mathcal{B}$.*

Proof. As in [Hov1][Proposition 5.1.4] one shows that cofibrations have the left lifting property with respect to trivial fibrations. Then it follows that if $A \rightarrow B$ is a map in $\mathcal{D}^{\mathcal{B}}$ with A cofibrant such that the maps $A_i \sqcup_{L_i A} L_i B \rightarrow B_i$ are (trivial) cofibrations then the map $\mathrm{colim} A \rightarrow \mathrm{colim} B$ is a (trivial) cofibration in \mathcal{D} . So a good trivial cofibration (definition as in the proof of [Hov1][Theorem 5.1.3]) with cofibrant domain is a trivial cofibration and trivial cofibrations with cofibrant domain have the left lifting property with respect to fibrations. We then can construct functorial factorizations into a good trivial cofibration followed by a fibration for maps with cofibrant domain as in the proof of [Hov1][Theorem 5.1.3])

and also the factorization into a cofibration followed by a trivial fibration (for the alternative definition for maps with cofibrant domain). It follows that a trivial cofibration with cofibrant domain is a good trivial cofibration. All other properties are immediate. \square

Similarly but easier we have that for an inverse category \mathcal{B} the diagram category $\mathcal{D}^{\mathcal{B}}$ is a J -semi model category.

We can combine both results as in [Hov1][Theorem 5.2.5] to get

Proposition 3. *Let \mathcal{D} be a J -semi model category and \mathcal{B} a Reedy category. Then $\mathcal{D}^{\mathcal{B}}$ is a J -semi model category where a map $f : A \rightarrow B$ is a weak equivalence if and only if it is objectwise a weak equivalence, a cofibration if and only if the maps $A_i \sqcup_{L_i A} L_i B \rightarrow B_i$ are cofibrations and a fibration if and only if the maps $A_i \rightarrow B_i \times_{M_i B} M_i A$ are fibrations.*

It is easily checked that cosimplicial and simplicial frames (see [Hov1][Definition 5.2.7]) exist on cofibrant objects. In the following we denote by A^\bullet and A_\bullet functorial cosimplicial and simplicial frames on cofibrant $A \in \mathcal{D}$. We are going to equip the category \mathcal{D}_{cf} with a strict 2-category structure $\mathcal{D}_{cf}^{\leq 2}$ with underlying 1-category \mathcal{D}_{cf} and with associated homotopy category $\text{Ho } \mathcal{D}_{cf}$. Let $A, B \in \mathcal{D}_{cf}$. As in [Hov1][Proposition 5.4.7] there are weak equivalences

$$\text{Hom}_{\mathcal{D}}(A^\bullet, B) \rightarrow \text{diag}(\text{Hom}_{\mathcal{D}}(A^\bullet, B_\bullet)) \leftarrow \text{Hom}_{\mathcal{D}}(A, B_\bullet)$$

in \mathbf{SSet} which are isomorphisms in degree 0, and we define the morphism category $\text{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(A, B)$ to be the groupoid associated to one of these simplicial sets. By the groupoid associated to a $K \in \mathbf{SSet}$ we mean the groupoid with set of objects $K[0]$ and set of morphisms $\text{Hom}(x, y)$ for $x, y \in K[0]$ the homotopy classes of paths from x to y in the topological realization of K . We have to give composition functors

$$\text{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(A, B) \times \text{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(B, C) \rightarrow \text{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(A, C).$$

These are the normal composition on objects and are induced on the morphisms by the map of simplicial sets

$$\text{Hom}_{\mathcal{D}}(A^\bullet, B) \times \text{Hom}_{\mathcal{D}}(B, C_\bullet) \rightarrow \text{diag}(\text{Hom}_{\mathcal{D}}(A^\bullet, C_\bullet)).$$

In the following we write \circ_0 for the composition of 2-morphisms over objects and \circ_1 for the composition of 2-morphisms over 1-morphisms. We claim that for $A, B, C \in \mathcal{D}_{cf}$, morphisms $f, g : A \rightarrow B$, $f', g' : B \rightarrow C$ and 2-morphisms $\varphi : f \rightarrow g$, $\psi : f' \rightarrow g'$ we have

$$\psi \circ_0 \varphi = (\text{Id}_{f'} \circ_0 \varphi) \circ_1 (\psi \circ_0 \text{Id}_g) = (\psi \circ_0 \text{Id}_f) \circ_1 (\text{Id}_{g'} \circ_0 \varphi).$$

This follows from the corresponding equation of homotopy classes of paths in $\text{Hom}_{\mathcal{D}}(A^\bullet, B) \times \text{Hom}_{\mathcal{D}}(B, C_\bullet)$. Moreover for a 1-morphism $f'' : C \rightarrow D$ we have $(\text{Id}_{f''} \circ_0 \psi) \circ_0 \text{Id}_f = \text{Id}_{f''} \circ_0 (\psi \circ_0 \text{Id}_f)$, and the assignments $\text{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(B, C) \rightarrow \text{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(B, D)$, $a \mapsto \text{Id}_{f''} \circ_0 a$, and $\text{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(B, C) \rightarrow \text{Hom}_{\mathcal{D}_{cf}^{\leq 2}}(A, C)$, $a \mapsto a \circ_0 \text{Id}_f$, are functors. From these three properties it follows that \circ_0 is associative and that \circ_0 and \circ_1 are compatible. Hence $\mathcal{D}_{cf}^{\leq 2}$ is a strict 2-category. We set $\text{Ho } \mathcal{D}^{\leq 2} := \mathcal{D}_{cf}^{\leq 2}$. One can show that this 2-category is weakly equivalent to the 2-truncation of the 1-Segal category (see [Si-Hi]) associated to \mathcal{D} .

Let \triangleleft be the category whose diagrams (i.e. functors into another category) are the “lower left triangles”, and \square the category whose diagrams are the commutative squares like the square at the beginning of this section. There is an obvious inclusion functor $\triangleleft \rightarrow \square$. For a category \mathcal{D} denote by $\mathcal{D}^{\triangleleft}$ (resp. \mathcal{D}^{\square}) the category of \triangleleft -diagrams (resp. of \square -diagrams) in \mathcal{D} . There is a restriction functor $r : \mathcal{D}^{\square} \rightarrow \mathcal{D}^{\triangleleft}$.

Let \mathcal{D} be a J -semi model category. Then there is a canonical way to define a *homotopy pushout functor*

$$\sqcup : (\mathrm{Ho} \mathcal{D})^{\triangleleft} \rightarrow (\mathrm{Ho} \mathcal{D})^{\square}$$

which sends a triangle B to the square $B \longrightarrow B \sqcup_A C$, together with a

$$\begin{array}{ccc} & B & \\ \uparrow & & \\ A & \longrightarrow & C \end{array} \quad \begin{array}{ccc} & B & \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

natural isomorphism from $r \circ \sqcup$ to the identity. This is done by lifting a triangle to a triangle in \mathcal{D} where all objects are cofibrant and at least one map is a cofibration. Then by the cube lemma ([Hov1][Lemma 5.2.6]), which is also valid for J -semi model categories, the pushout does not depend on the choices and indeed yields a well-defined square in $\mathrm{Ho} \mathcal{D}$. We call a square in $\mathrm{Ho} \mathcal{D}$ a *homotopy pushout square* if it is in the essential image of the functor \sqcup . This is by definition the same as to say that it is the image of a homotopy pushout square in \mathcal{D} , which is defined to be any commutative square weakly equivalent to a pushout square

$$\begin{array}{ccc} B & \longrightarrow & D \\ \uparrow f & & \uparrow \\ A & \xrightarrow{g} & C \end{array}$$

where all objects are cofibrant and f or g is a cofibration.

Taking A to be an initial object in $\mathrm{Ho} \mathcal{D}$ (i.e. the image of an initial object in \mathcal{D}) the product \sqcup_A gives the categorical coproduct on $\mathrm{Ho} \mathcal{D}$. For general A the homotopy pushout need not be a categorical pushout in $\mathrm{Ho} \mathcal{D}$.

We show that the homotopy pushout *has* a categorical interpretation in the 2-category $\mathrm{Ho}^{\leq 2} \mathcal{D}$: Let for the moment \mathcal{D} be an arbitrary 2-category. A commutative square

$$\begin{array}{ccc} B & \xrightarrow{g'} & D \\ \uparrow f & \searrow \varphi & \uparrow f' \\ A & \xrightarrow{g} & C \end{array}$$

in \mathcal{D} is called a *homotopy pushout*, if for any object $T \in \mathcal{D}$ the square

$$\begin{array}{ccc} \mathrm{Hom}(D, T) & \longrightarrow & \mathrm{Hom}(B, T) \\ \downarrow & \swarrow & \downarrow \\ \mathrm{Hom}(C, T) & \longrightarrow & \mathrm{Hom}(A, T) \end{array}$$

is a homotopy pullback in the 2-category \mathbf{Gpd} of small groupoids. We recall the definition of a homotopy pullback in \mathbf{Gpd} : For a triangle $K \xrightarrow{f} G \xleftarrow{g} L$ in \mathbf{Gpd} we define the *homotopy fibre product* $K \times_G^h L$ to be the groupoid with objects

triples (x, y, φ) , where $x \in K$, $y \in L$ and $\varphi : f(x) \xrightarrow{\cong} g(y)$ an isomorphism, and morphisms $(x, y, \varphi) \rightarrow (x', y', \varphi')$ pairs of morphisms $x \rightarrow x'$, $y \rightarrow y'$ making the obvious diagram in G commutative. Now for a commutative square

$$\begin{array}{ccc} M & \longrightarrow & L \\ \downarrow & \swarrow & \downarrow \\ K & \longrightarrow & G \end{array}$$

in **Gpd** there is a canonical functor $M \rightarrow K \times_G^h L$, and we say that the square is a homotopy pullback if this functor is an equivalence.

Let \mathcal{D} be again a J -semi model category. We claim now that the image of a homotopy pushout square

$$\begin{array}{ccc} B & \longrightarrow & D \\ \uparrow f & & \uparrow \\ A & \xrightarrow{g} & C \end{array}$$

in \mathcal{D}_{cf} in $\text{Ho}^{\leq 2}\mathcal{D}$ is a homotopy pushout square in the sense just defined: So let $T \in \mathcal{D}_{cf}$. Then $\text{Hom}(\text{square}, T_\bullet)$ is a homotopy pullback square in **SSet**, since, if f is a cofibration with A cofibrant, the map $\text{Hom}(f, T_\bullet)$ is a fibration in **SSet**. As is easily verified the functor **SSet** \rightarrow **Gpd** preserves homotopy pullbacks, hence our claim follows.

So we have shown the following: $\text{Ho}^{\leq 2}\mathcal{D}$ has categorical homotopy pushouts, every homotopy pushout square in $\text{Ho } \mathcal{D}$ comes from one in $\text{Ho}^{\leq 2}\mathcal{D}$, every homotopy pushout square in $\text{Ho}^{\leq 2}\mathcal{D}$ is equivalent to the image of a homotopy pushout square in \mathcal{D} and all such images are homotopy pushout squares in $\text{Ho}^{\leq 2}\mathcal{D}$.

Note that it follows that for any $T \in \text{Ho } \mathcal{D}$ and homotopy pushout square as above the map

$$\text{Hom}(B \sqcup_A C, T) \rightarrow \text{Hom}(B, T) \times_{\text{Hom}(A, T)} \text{Hom}(C, T),$$

where all homomorphism sets are in $\text{Ho } \mathcal{D}$, is always surjective.

There is a dual *homotopy pullback functor* \times and the dual notion of a homotopy pullback square in both $\text{Ho } \mathcal{D}$ and $\text{Ho}^{\leq 2}\mathcal{D}$.

For a cofibrant object $A \in \mathcal{D}$ the category $A \downarrow \mathcal{D}$ of objects under A is again a J -semi model category. The 2-functor

$$\mathcal{D} \rightarrow \mathbf{Cat},$$

$$A \mapsto \text{Ho}((QA) \downarrow \mathcal{D})$$

where $QA \rightarrow A$ is a cofibrant replacement, descends to a 2-functor

$$\text{Ho}^{\leq 2}\mathcal{D} \rightarrow \mathbf{Cat},$$

$$A \mapsto D(A \downarrow \mathcal{D})$$

such that the image functors f_* of all maps f in $\text{Ho}^{\leq 2}\mathcal{D}$ have right adjoints f^* . The functor f_* preserves homotopy pushout squares, and the functor f^* preserves homotopy pullback and homotopy pushout squares. For $f : 0 \rightarrow A$ the map from an initial object to an object in $\text{Ho}^{\leq 2}\mathcal{D}$ the functor $f^* : D(A \downarrow \mathcal{D}) \rightarrow \text{Ho } \mathcal{D}$ factors through $A \downarrow \text{Ho } \mathcal{D}$ and the map from A to the image of the initial object in $D(A \downarrow \mathcal{D})$ is an isomorphism.

Consider a commutative square

$$\begin{array}{ccc} B & \xrightarrow{g'} & D \\ \uparrow f & \searrow \varphi & \uparrow f' \\ A & \xrightarrow{g} & C \end{array}$$

in $\mathrm{Ho}^{\leq 2}\mathcal{D}$. Let $E \in D(B \downarrow \mathcal{D})$. There is a base change morphism

$$g_* f^* E \rightarrow f'^* g'_* E$$

adjoint to the natural map $f^* E \rightarrow f^* g'^* g'_* M \xrightarrow{\varphi} g^* f'^* g'_* M$. This base change morphism applied to diagrams

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & \searrow & \uparrow \\ A & \xrightarrow{\mathrm{Id}} & A \end{array}$$

enables one to construct a 2-functor

$$(A \downarrow \mathrm{Ho}^{\leq 2}\mathcal{D}) \rightarrow D(A \downarrow \mathcal{D})$$

which gives an equivalence after 1-truncation of the left hand side.

Remark 1. *The above construction should generalize to give functors between (weak) $(n+1)$ -categories*

$$\begin{aligned} \mathrm{Ho}^{\leq n+1}\mathcal{D} &\rightarrow n - \mathbf{Cat} \\ A &\mapsto D^{\leq n}(A \downarrow \mathcal{D}), \end{aligned}$$

where $\mathrm{Ho}^{\leq n+1}$ is the $(n+1)$ -truncation of the 1-Segal category associated to \mathcal{D} , $n - \mathbf{Cat}$ is the $(n+1)$ -category of n -categories and $D^{\leq n}(A \downarrow \mathcal{D}) := \mathrm{Ho}^{\leq n}(QA \downarrow \mathcal{D})$ for $QA \rightarrow A$ a cofibrant replacement.

There are dual constructions for objects over an object in \mathcal{D} .

The following theorem is the main source to obtain J -semi model categories.

Theorem 2. *Let \mathbb{T} be a monad in \mathcal{C} and assume that $\mathcal{C}[\mathbb{T}]$ has coequalizers. Suppose that every map in $\mathbb{T}J$ -cell whose domain is cofibrant in \mathcal{C} is a weak equivalence in \mathcal{C} and every map in $\mathbb{T}I$ -cell whose domain is cofibrant in \mathcal{C} is a cofibration in \mathcal{C} (here in both cases the cell complexes are built in $\mathcal{C}[\mathbb{T}]$). Assume furthermore that the initial object in $\mathcal{C}[\mathbb{T}]$ is cofibrant in \mathcal{C} . Then there is a cofibrantly generated J -semi model structure on $\mathcal{C}[\mathbb{T}]$ over \mathcal{C} , where a map is a weak equivalence or fibration if and only if it is a weak equivalence or fibration in \mathcal{C} .*

Proof. We define the weak equivalences (resp. fibrations) as the maps in $\mathcal{C}[\mathbb{T}]$ which are weak equivalences (resp. fibrations) as maps in \mathcal{C} . By adjointness the fibrations are $\mathbb{T}J$ -inj and the trivial fibrations are $\mathbb{T}I$ -inj. We define the class of cofibrations to be $\mathbb{T}I$ -cof. Since the adjoint of \mathbb{T} is the forgetful functor property 1 of Definition 1 is clear.

The bicompleteness of $\mathcal{C}[\mathbb{T}]$ follows as in the proof of Theorem 1. The 2-out-of-3 and retract axioms for the weak equivalences and the fibrations hold in $\mathcal{C}[\mathbb{T}]$ since they hold in \mathcal{C} , the retract axiom for the cofibrations holds because $\mathbb{T}I$ -cof is closed under retracts. So property 2 is fulfilled.

The first half of property 3 is true by the definition of the cofibrations. By our smallness assumptions we have functorial factorizations of maps into a cofibration followed by a trivial fibration and into a map from $\mathbb{T}J$ -cell followed by a fibration. We claim that a map f in $\mathbb{T}J$ -cell whose domain is cofibrant in \mathcal{C} is a trivial cofibration. f is a weak equivalence by assumption. Factor f as $p \circ i$ into a cofibration followed by a trivial fibration. Since f has the left lifting property with respect to p , f is a retract of i by the retract argument, hence also a cofibration. Hence we have shown property 4.

Now let f be a trivial cofibration whose domain is cofibrant in \mathcal{C} . We can factor f as $p \circ i$ with $i \in \mathbb{T}J$ -cell and p a fibration. p is a trivial fibration by the 2-out-of-3 property, hence f has the left lifting property with respect to p , so f is a retract of i and has therefore the left lifting property with respect to fibrations. This is the second half of property 3. Property 5 immediately follows from the assumptions, and property 6 is true since limits in $\mathcal{C}[\mathbb{T}]$ are computed in \mathcal{C} . \square

Alternative: Assume that $\mathcal{C}[\mathbb{T}]$ has coequalizers, that sequential colimits in $\mathcal{C}[\mathbb{T}]$ are computed in \mathcal{C} and that the pushout of an object in $\mathcal{C}[\mathbb{T}]$ which is cofibrant in \mathcal{C} by a map from $\mathbb{T}I$ (resp. from $\mathbb{T}J$) is a cofibration (resp. weak equivalence) as a map in \mathcal{C} . Then the same conclusion holds as in the Theorem above. Moreover the conclusion also holds for the alternative definition of J -semi model category without the smallness assumptions on the domains of I and J which we made at the beginning of this section.

Example 1. Let $\text{Ass}(\mathcal{C})$ be the category of associative unital algebras in \mathcal{C} . Then $\text{Ass}(\mathcal{C})$ is a J -semi model category over \mathcal{C} (see [Hov2, Theorem 3.3]).

Will will need the

Lemma 2. Let R be a ring with unit in \mathcal{C} , i a map in $(I \otimes R)\text{-cof}$ (taken in $R\text{-Mod}_r$) and j a map in $R\text{-Mod}$ which is a (trivial) cofibration in \mathcal{C} . Then $i \square_{Rj}$ is a (trivial) cofibration in \mathcal{C} . If i is in $(J \otimes R)\text{-cof}$, then $i \square_{Rj}$ is a trivial cofibration in \mathcal{C} .

Proof. This follows either by [Hov1][Lemma 4.2.4] applied to the adjunction of two variables $R\text{-Mod}_r \times R\text{-Mod} \rightarrow \mathcal{C}$, $(M, N) \mapsto M \otimes_R N$, or by Lemma 1. \square

3. OPERADS

For a group G write $\mathcal{C}[G]$ for the category of objects in \mathcal{C} together with a right G -action. This is the same as $\mathbb{1}[G]\text{-Mod}_r$, where $\mathbb{1}[G]$ is the group ring of G in \mathcal{C} . Let $\mathcal{C}^{\mathbb{N}}$ be the category of sequences in \mathcal{C} and \mathcal{C}^{Σ} the category of symmetric sequences, i.e. $\mathcal{C}^{\Sigma} = \coprod_{n \in \mathbb{N}} \mathcal{C}[\Sigma_n]$. Finally let $\mathcal{C}^{\mathbb{N}, \bullet}$ (resp. $\mathcal{C}^{\Sigma, \bullet}$) be the category of objects X from $\mathcal{C}^{\mathbb{N}}$ (resp. from \mathcal{C}^{Σ}) together with a map $\mathbb{1} \rightarrow X(1)$.

Proposition 4. For any group G the category $\mathcal{C}[G]$ has a natural structure of cofibrantly generated model category with generating cofibrations $I[G]$ and generating trivial cofibrations $J[G]$.

Proof. Easy from [Hov1][Theorem 2.1.19]. \square

Hence there are also canonical model structures on $\mathcal{C}^{\mathbb{N}}$, $\mathcal{C}^{\mathbb{N}, \bullet}$, \mathcal{C}^{Σ} and $\mathcal{C}^{\Sigma, \bullet}$.

Note that a map of groups $\varphi : H \rightarrow G$ induces a left Quillen functor $\mathcal{C}[H] \rightarrow \mathcal{C}[G]$. If φ is injective the right adjoint to this functor preserves (trivial) cofibrations.

Let $\text{Op}(\mathcal{C})$ be the category of operads in \mathcal{C} , where an operad in \mathcal{C} is defined as in [KM][Definition 1.1]. Let $F : \mathcal{C}^{\mathbb{N}} \rightarrow \text{Op}(\mathcal{C})$ be the functor which assigns to a sequence X the free operad FX on X . This functor naturally factors through $\mathcal{C}^{\mathbb{N}, \bullet}$, \mathcal{C}^{Σ} and $\mathcal{C}^{\Sigma, \bullet}$, and the functors starting from one of these categories going to $\text{Op}(\mathcal{C})$ are also denoted by F . The right adjoints of F , i.e. the forgetful functors, map \mathcal{O} to $\mathcal{O}^{\#}$.

For any object $A \in \mathcal{C}$ there is the endomorphism operad $\text{End}^{\text{Op}}(A)$ given by $\text{End}^{\text{Op}}(A)(n) = \underline{\text{Hom}}(A^{\otimes n}, A)$.

We come to the main result of this section:

Theorem 3. *The category $\text{Op}(\mathcal{C})$ is a cofibrantly generated J -semi model category over $\mathcal{C}^{\Sigma, \bullet}$ with generating cofibrations FI and generating trivial cofibrations FJ . If \mathcal{C} is left proper (resp. right proper), then $\text{Op}(\mathcal{C})$ is left proper relative to $\mathcal{C}^{\Sigma, \bullet}$ (resp. right proper).*

We first give an explicit description of free operads and pushouts by free operad maps, which will be needed for the proof of this Theorem.

- Definition 3.**
1. *An n -tree is a finite connected directed graph T such that any vertex of T has ≤ 1 ingoing arrows, the outgoing arrows of each vertex v of T are numbered by $1, \dots, \text{val}(v)$, where $\text{val}(v)$ is the number of these arrows, and there are n arrows which do not end at any vertex, which are called tails and which are numbered by $1, \dots, n$. By definition the empty tree has one tail, so it is a 1-tree.*
 2. *A doubly colored n -tree is an n -tree together with a decomposition of the set of vertices into old and new vertices.*
 3. *A proper doubly colored n -tree is a doubly colored n -tree such that every arrow starting from an old vertex is either a tail or goes to a new vertex.*

We denote the set of n -trees by $\mathcal{T}(n)$, the set of doubly colored n -trees by $\mathcal{T}_{\text{dc}}(n)$ and the set of proper doubly colored n -trees by $\mathcal{T}_{\text{dc}}^p(n)$. Set $\mathcal{T} := \coprod_{n \in \mathbb{N}} \mathcal{T}(n)$ and $\mathcal{T}_{\text{dc}}^{(p)} := \coprod_{n \in \mathbb{N}} \mathcal{T}_{\text{dc}}^{(p)}(n)$.

The n -trees will describe the n -ary operations of free operads, and indeed $\mathcal{T}(\bullet)$ is endowed with a natural operad structure in **Set**. Let $n, m_1, \dots, m_n \in \mathbb{N}$, $m := \sum_{i=1}^n m_i$ and $T \in \mathcal{T}(n)$, $T_i \in \mathcal{T}(m_i)$, $i = 1, \dots, n$. Then the corresponding structure map γ of this operad sends (T, T_1, \dots, T_n) to the tree which one obtains from T by glueing the root of T_i to the i -th tail of T for every $i = 1, \dots, n$. The previously j -th tail of T_i gets the label $j + \sum_{k=1}^{i-1} m_k$. The free right action of Σ_n on $\mathcal{T}(n)$ (which is also defined on $\mathcal{T}_{\text{dc}}^{(p)}(n)$) is such that $\sigma \in \Sigma$ sends a tree $T \in \mathcal{T}(n)$ to the tree obtained from T by changing the label i of a tail of T into $\sigma^{-1}(i)$. So $\gamma(T, T_1, \dots, T_n)^{\sigma(m_{\sigma(1)}, \dots, m_{\sigma(n)})} = \gamma(T^{\sigma}, T_{\sigma(1)}, \dots, T_{\sigma(n)})$, where $\sigma(m_1, \dots, m_n)$ permutes blocks of length m_i in $1, \dots, m$ as σ permutes $1, \dots, n$.

Note that an n -tree has a natural embedding into the plane and this embedding is equivalent to the numbering of the arrows. It follows that there exists a *canonical*

labelling of the tails of an n -tree, namely the one which labels the tails successively from the left to the right in the planar embedding of the tree.

For T an element of \mathcal{T} or $\mathcal{T}_{\text{dc}}^{(p)}$ let $V(T)$ denote the set of vertices of T (this is defined up to unique isomorphism, since our trees do not have automorphisms) and let $u(T)$ be the number of vertices of T of valency 1 and $U(T)$ be the set of vertices of T of valency 1. For $T \in \mathcal{T}_{\text{dc}}^{(p)}$ write $V_{\text{old}}(T)$ (resp. $V_{\text{new}}(T)$) for the set of old (resp. new) vertices of T and $U_{\text{old}}(T)$ (resp. $U_{\text{new}}(T)$) for the set of old (resp. new) vertices in $U(T)$ and $u_{\text{old}}(T)$ (resp. $u_{\text{new}}(T)$) for their number.

Proposition 5. 1. *The free operad FX on $X \in \mathcal{C}^{\mathbb{N}}$ is given by*

$$(FX)(n) = \coprod_{T \in \mathcal{T}(n)} \bigotimes_{v \in V(T)} X(\text{val}(v)) .$$

2. *The free operad FX on $X \in \mathcal{C}^{\mathbb{N}, \bullet}$ is given by a ω -sequence*

$$FX = \text{colim}_{i < \omega} F_i X$$

in $\mathcal{C}^{\mathbb{N}}$, where $(F_i X)_n$ is a pushout of $(F_{i-1} X)_n$ by the map

$$\coprod_{\substack{T \in \mathcal{T}(n) \\ u(T) = i}} \left(\bigotimes_{v \in V(T) \setminus U(T)} X(\text{val}(v)) \right) \otimes e^{\square(U(T))} ,$$

where e is the unit map $\mathbb{1} \rightarrow X(1)$.

3. *The free operad on $X \in \mathcal{C}^{\Sigma}$ is given by*

$$(FX)(n) = \left(\coprod_{T \in \mathcal{T}(n)} \bigotimes_{v \in V(T)} X(\text{val}(v)) \right) / \sim ,$$

where the equivalence relation \sim identifies for every isomorphism of directed graphs $\varphi : T \rightarrow T'$, $T, T' \in \mathcal{T}(n)$, which respects the numbering of the tails but not necessarily of the arrows, the summands $\bigotimes_{v \in V(T)} X(\text{val}(v))$ and $\bigotimes_{v \in V(T')} X(\text{val}(v))$ by the map $\bigotimes_{v \in V(T)} \sigma_v$, where $\sigma_v : X(\text{val}(v)) \rightarrow X(\text{val}(\varphi(v))) = X(\text{val}(v))$ is the action of the element $\sigma_v \in \Sigma_{\text{val}(v)}$ such that φ maps the i -th arrow of v to the $\sigma_v(i)$ -th arrow of $\varphi(v)$.

4. *The free operad FX on $X \in \mathcal{C}^{\Sigma, \bullet}$ is given by a ω -sequence*

$$FX = \text{colim}_{i < \omega} F_i X$$

in $\mathcal{C}^{\mathbb{N}}$, where $(F_i X)_n$ is a pushout of $(F_{i-1} X)_n$ by the map

$$\left(\coprod_{T \in \mathcal{T}(n), u(T)=i} \left(\bigotimes_{v \in V(T) \setminus U(T)} X(\text{val}(v)) \right) \otimes e^{\square(U(T))} \right) / \sim ,$$

where e is as in 2 and the equivalence relation \sim is like in 3.

In cases 2 and 4 the attaching map is induced from the operation of removing a vertex of valency 1 from a tree. Note that the morphism in 4 and the attaching morphism respects the equivalence relation. The Σ_n -actions are induced from the Σ_n -action on $\mathcal{T}(n)$.

Proof. We claim that in all four cases the functors F define a monad the algebras of which are the operads in \mathcal{C} . So we have to define in all four cases maps $m : FFX \rightarrow FX$ and $e : X \rightarrow FX$ satisfying the axioms for a monad. We will restrict ourselves to case i) and leave the other cases to the interested reader.

The domain of the map $m(n)$ is a coproduct over all $T \in \mathcal{T}(n)$, $T_v \in \mathcal{T}(\text{val}(v))$ for all $v \in V(T)$ over the

$$\bigotimes_{v \in V(T), w \in V(T_v)} X(\text{val}(w)) ,$$

and the map m sends such an entry via the identity to the entry associated to the tree in $\mathcal{T}(n)$ obtained by replacing every vertex v of T by the tree T_v in such a way that the numbering of the arrows starting at v and the numbering of the tails of T_v correspond. The map e sends $X(n)$ to the summand $X(n)$ in FX which belongs to the tree with one vertex and n tails such that the labelling of the arrows coincides with the labelling of the tails (which are of course all arrows in this case) (i.e. the labelling of the tails is the canonical one). It is clear that m is associative and e is a two-sided unit. To see that an F -algebra is the same as an operad one proceeds as follows: Let X be an F -algebra. Let $\mathcal{O}(n) := X(n)$. The structure maps of the operad structure we will define on \mathcal{O} are obtained from the algebra map by restricting it to the summands belonging to trees where every arrow starting at the root goes to a vertex which has only tails as outgoing vertices and where the labelling of the tails is the canonical one. The unit in $\mathcal{O}(1)$ corresponds to the empty tree. The right action of a $\sigma \in \Sigma_n$ on $\mathcal{O}(n)$ is given by the algebra map restricted to the tree with one vertex and n tails such that the i -th arrow simultaneously is the $\sigma^{-1}(i)$ -th tail. That 1 acts as the identity is the unit property of X , and the associativity of the action follows from the associativity of X . It is easy to see that the associativity and symmetry properties of \mathcal{O} also follow from the associativity of X . The unit properties follow from the behaviour of the empty tree.

On the other hand let \mathcal{O} be an operad. We define an F -algebra structure on $X := \mathcal{O}^\sharp$: Let $T \in \mathcal{T}(n)$ be a tree with canonical labelling of the tails. Then it is clear how to define a map from the summand in FX corresponding to T to $X(n)$ by iterated application of the structure maps of \mathcal{O} (the unit of \mathcal{O} is needed to get the map for the empty tree). The map on the summand corresponding to T^σ for $\sigma \in \Sigma_n$ is the map for T followed by the action of σ on $X(n) = \mathcal{O}(n)$. One then can check that the associativity, symmetry and unit properties of the structure maps of \mathcal{O} imply that we get indeed an F -algebra with structure map $FX \rightarrow X$ just described. \square

For describing pushouts by free operad maps we need an operation which changes a new vertex in a tree in $\mathcal{T}_{\text{dc}}^p(n)$ into an old vertex and gives again a tree in $\mathcal{T}_{\text{dc}}^p(n)$. This is given by first making the new vertex into an old vertex to get an element of $\mathcal{T}_{\text{dc}}(n)$ and then removing all arrows joining only old vertices and identifying the old vertices which have been joined. The numbering of the arrows of the new tree is most easily described by noting that this numbering corresponds to a planar embedding of the tree and the operation of removing the arrows and identifying the vertices can canonically be done in the plane. For $T \in \mathcal{T}_{\text{dc}}^p$ and $v \in V_{\text{new}}(T)$ denote by $\text{ch}_T(v) \in \mathcal{T}_{\text{dc}}^p$ the tree obtained by changing the new vertex v in T into

an old vertex. Note that for $\mathcal{O} \in \text{Op}(\mathcal{C})$ there is a concatenation map

$$\text{conc}_T^{\mathcal{O}}(v) : \mathcal{O}(\text{val}(v)) \otimes \bigotimes_{v' \in V_{\text{old}}(V)} \mathcal{O}(\text{val}(v')) \longrightarrow \bigotimes_{v' \in V_{\text{old}}(\text{ch}_T(v))} \mathcal{O}(\text{val}(v'))$$

induced by applying the operad maps of \mathcal{O} .

Proposition 6. *Let $\mathcal{O} \in \text{Op}(\mathcal{C})$ and $f : A \rightarrow B$ and $\varphi : A \rightarrow \mathcal{O}^\sharp$ be maps in $\mathcal{C}^\mathbb{N}$. Then the pushout \mathcal{O}' of \mathcal{O} by Ff with attaching map the adjoint of φ is given by a $\omega \times (\omega + 1)$ -sequence $\mathcal{O}' = \text{colim}_{(i,j) < \omega \times (\omega + 1)} \mathcal{O}_{(i,j)}$ in $\mathcal{C}^\mathbb{N}$, where for $j < \omega$ $\mathcal{O}_{(i,j)}(n)$ is a pushout of $\mathcal{O}_{(i,j)-1}(n)$ in \mathcal{C} by the quotient of the map*

$$\coprod \left(\bigotimes_{v \in V_{\text{old}}(T) \setminus U_{\text{old}}(T)} \mathcal{O}(\text{val}(v)) \right) \otimes e^{\square(U_{\text{old}}(T))} \square_{v \in V_{\text{new}}(T)} f(\text{val}(v)) ,$$

where the coproduct is over all $T \in \mathcal{T}_{\text{dc}}^p(n)$ with $\sharp V_{\text{new}}(T) = i$ and $u_{\text{old}}(T) = j$, with respect to the equivalence relation which identifies for every isomorphism of doubly colored directed graphs $\varphi : T \rightarrow T'$, $T, T' \in \mathcal{T}_{\text{dc}}^p$, which respects the labeling of the tails and of the arrows starting at new vertices, the summands corresponding to T and T' via a map analogous to the map in Proposition 5.3. Here e is the unit $\mathbb{1} \rightarrow \mathcal{O}(1)$ and the attaching map is the following: The domain of the above map is obtained by glueing $i + j$ objects together, hence we have to give $i + j$ maps compatible with glueing. The first i maps are induced by removing one of the vertices in $U_{\text{old}}(T)$ from T , and the other j maps are induced by changing one of the vertices in $V_{\text{new}}(T)$ into an old vertex and applying the maps $\text{conc}_v^{\mathcal{O}}(T)$, $v \in V_{\text{new}}(T)$. (Note that for $n = 1$ the operad \mathcal{O} appears in the second step of the limit, in all other cases in the first.) The Σ_n -actions are induced from the ones on $\mathcal{T}_{\text{dc}}^p(n)$.

There are similar descriptions of pushouts of \mathcal{O} by free operad maps on maps from $\mathcal{C}^{\mathbb{N}, \bullet}$, \mathcal{C}^Σ and $\mathcal{C}^{\Sigma, \bullet}$.

Proof. Let $\tilde{\mathcal{O}}(n)$ be the colimits described in the Proposition. First of all we check that this is well defined, i.e. that firstly the $i + j$ maps we have described glue together. This is the case because the processes of removing old vertices of valency 1 and/or changing a new vertex into an old one and concatenating commute with each other. Secondly this map factors through the quotient described in the Proposition because of the symmetry properties of \mathcal{O} and because of the fact that in previous steps quotients with respect to analogous equivalence relations have been taken.

Next we have to equip $\tilde{\mathcal{O}} \in \mathcal{C}^\mathbb{N}$ with an operad structure. The unit is the one coming from \mathcal{O} . We define the structure map $\gamma : \tilde{\mathcal{O}}(n) \otimes \tilde{\mathcal{O}}(m_1) \otimes \cdots \otimes \tilde{\mathcal{O}}(m_n) \rightarrow \tilde{\mathcal{O}}(m)$ ($m = \sum_{i=1}^n m_i$) in the following way: For $T \in \mathcal{T}_{\text{dc}}^p(n)$ let

$$S(T) := \left(\bigotimes_{v \in V_{\text{old}}(T)} \mathcal{O}(\text{val}(v)) \right) \otimes \bigotimes_{v \in V_{\text{new}}(T)} B(\text{val}(v)) .$$

First one defines for trees $T \in \mathcal{T}_{\text{dc}}^p(n)$, $T_i \in \mathcal{T}_{\text{dc}}^p(m_i)$, $i = 1, \dots, n$, a map

$$\xi_{(T, T_1, \dots, T_n)} : S(T) \otimes S(T_1) \otimes \cdots \otimes S(T_n) \rightarrow \tilde{\mathcal{O}}(m) .$$

Therefore one glues the tree T_i to the tail of T with label i and concatenates such that one gets a tree $\tilde{T} \in \mathcal{T}_{\text{dc}}^p(m)$. Then by applying structure maps of \mathcal{O} one gets

a map $S(T) \otimes S(T_1) \otimes \cdots \otimes S(T_n) \rightarrow S(\tilde{T})$ and composes this with the canonical map $S(\tilde{T}) \rightarrow \tilde{\mathcal{O}}(m)$.

Let $m_0 := n$. Suppose we have already defined for a $0 \leq k \leq n$ and for all trees $T_i \in \mathcal{T}_{\text{dc}}^p(m_i)$, $i = k, \dots, n$, a map $(\bigotimes_{i=0}^{k-1} \tilde{\mathcal{O}}(m_i)) \otimes S(T_k) \otimes \cdots \otimes S(T_n) \rightarrow \tilde{\mathcal{O}}(m)$. From this data one then obtains the same data for $k+1$ instead of k as follows: Let $T_i \in \mathcal{T}_{\text{dc}}^p(m_i)$, $i = k+1, \dots, n$. One defines the map $\varphi : (\bigotimes_{i=0}^k \tilde{\mathcal{O}}(m_i)) \otimes S(T_{k+1}) \otimes \cdots \otimes S(T_n) \rightarrow \tilde{\mathcal{O}}(m)$ by transfinite induction on the terms of the $\omega \times (\omega+1)$ -sequence defining $\tilde{\mathcal{O}}(m_k)$: So let $(\bigotimes_{i=0}^{k-1} \tilde{\mathcal{O}}(m_i)) \otimes \mathcal{O}_{(i,j)}(m_k) \otimes S(T_k) \otimes \cdots \otimes S(T_n) \rightarrow \tilde{\mathcal{O}}(m)$ be already defined and let ψ be the map by which $\mathcal{O}_{(i,j+1)}(m_k)$ is a pushout of $\mathcal{O}_{(i,j)}(m_k)$. We define φ on $(\bigotimes_{i=0}^{k-1} \tilde{\mathcal{O}}(m_i)) \otimes \mathcal{O}_{(i,j+1)}(m_k) \otimes S(T_k) \otimes \cdots \otimes S(T_n)$ by using the data described above for k to get the map after taking the appropriate quotient on the codomain of ψ . One has to check the compatibility of this map with the given map via the attaching map. To do this for one of the $i+j$ summands of the domain of ψ one uses the fact that the same kind of compatibility is valid in $\tilde{\mathcal{O}}(m)$. Finally when arriving at $k = n$ we get the desired structure map.

The associativity of the structure maps follows by proving the corresponding statement for the $\xi_{(T, T_1, \dots, T_n)}$. This one gets by first glueing trees without concatenating and then observing that the concatenation processes at different places commute. The symmetry properties follow in the same way as for free operads, the unit properties are forced by the fact that in the ψ 's the pushout product over the unit maps is taken. Hence $\tilde{\mathcal{O}}$ is an operad. It receives canonical compatible maps in $\text{Op}(\mathcal{C})$ from \mathcal{O} and FB .

In the end we have to show that our operad $\tilde{\mathcal{O}}$ indeed satisfies the universal property of the pushout by Ff . We need to show that a map $g : \mathcal{O} \rightarrow \mathcal{O}''$ in $\text{Op}(\mathcal{C})$ together with a map $h : B \rightarrow (\mathcal{O}'')^\#$ compatible with the attaching map is the same as a map $g' : \tilde{\mathcal{O}} \rightarrow \mathcal{O}''$. To get g' from g and h one first defines for any $T \in \mathcal{T}_{\text{dc}}^p(n)$ a map $S(T) \rightarrow \mathcal{O}''$ using the structure maps of \mathcal{O}'' . Then one checks that these maps indeed glue together to a g' . To get g and h from g' one composes g' with $\mathcal{O} \rightarrow \tilde{\mathcal{O}}$ and $B \rightarrow FB \rightarrow \tilde{\mathcal{O}}$. These processes are inverses to each other. \square

Lemma 3. *Let \mathcal{O} , f , φ and \mathcal{O}' be as in Proposition 6, assume that $\mathcal{O} \in \text{Op}(\mathcal{C})$ is cofibrant as object in $\mathcal{C}^{\Sigma, \bullet}$ and that f is a (trivial) cofibration. Then the pushout $\mathcal{O} \rightarrow \mathcal{O}'$ is a (trivial) cofibration in $\mathcal{C}^{\Sigma, \bullet}$. There is an analogous statement for f a (trivial) cofibration in $\mathcal{C}^{\mathbb{N}, \bullet}$, \mathcal{C}^{Σ} and $\mathcal{C}^{\Sigma, \bullet}$.*

In the following Lemma we use the fact that if we have a G -action on an object L and a Σ_n -action on M , then there is a canonical action of the wreath product $\Sigma_n \ltimes G^n$ on $M \otimes L^{\otimes n}$.

Lemma 4. *Let $n_1, \dots, n_k \in \mathbb{N}_{>0}$, let G_1, \dots, G_k be groups and g_i be a cofibration in $\mathcal{C}[G_i]$, $i = 1, \dots, k$. Let f be a cofibration in $\mathcal{C}[\prod_{i=1}^k \Sigma_{n_i}]$. Then the map*

$$h := f \square \prod_{i=1}^k g_i^{\square n_i}$$

is a cofibration in $\mathcal{C}[(\prod_{i=1}^k \Sigma_{n_i}) \ltimes (\prod_{i=1}^k G_i^{n_i})]$. If f or one of the g_i is trivial, so is h .

Proof. We restrict to the case $k = 1$, the general case is done in the same way. Set $n := n_1$, $G := G_1$ and $g := g_1$. We can assume that $g \in I[G]$ -cell and

$f \in I[\Sigma_n]$ -cell (or $f \in J[G]$ -cell or $g \in J[\Sigma_n]$ -cell). Let $g : L_0 \rightarrow \operatorname{colim}_{i < \mu} L_i$ and $f : M_0 \rightarrow \operatorname{colim}_{i < \lambda} M_i$ such that $L_i \rightarrow L_{i+1}$ is a pushout by $\psi_i \in I[G]$ and $M_i \rightarrow M_{i+1}$ is a pushout by $\varphi_i \in I[\Sigma_n]$. Then by Lemma 1 $f \square g^{\square n}$ is a $\lambda \times \mu^n$ -sequence, and the transition maps are pushouts by the $\varphi_i \square \psi_{i_1} \square \cdots \square \psi_{i_n}$, $i < \lambda$; $i_1, \dots, i_n < \mu$. We can modify this sequence to make it invariant under the Σ_n -action: Let S be the set of unordered sequences of length n with entries in μ , and for $s \in S$ let j_s be the set of ordered sequences of length n with entries in μ which map to s . Let $s, s' \in S$. In the following let us view s and s' as monotonically increasing sequences of length n . We say that $s < s'$ if there is a $1 \leq i < n$ such that $s(j) = s'(j)$ for $i < j$ and $s(i) < s'(i)$. With this order S is well-ordered. Now $g^{\square n}$ is an S -sequence with s -th transition map $\psi'_s := \coprod_{w \in j_s} \psi_{w(1)} \square \cdots \square \psi_{w(n)}$, so $f \square g^{\square n}$ is the corresponding $\lambda \times S$ -sequence with transition maps the $\varphi_i \square \psi'_s$, $i < \lambda$, $s \in S$. Note that on these maps there is a $\Sigma_n \ltimes G^n$ -action. Now to prove our claim it suffices to show that every $\varphi_i \square \psi'_s$ is a (trivial) cofibration in $\mathcal{C}[\Sigma_n \ltimes G^n]$, which can easily be seen by noting that every φ_i and ψ_i is of the form $h[G]$ for $h \in I$ (or $h \in J$). \square

Proof of Lemma 3. Let \sim be the equivalence relation on $\mathcal{T}_{\text{dc}}^p$ which identifies T and T' in $\mathcal{T}_{\text{dc}}^p$ if there is an isomorphism of directed graphs $T \rightarrow T'$ which respects the labeling of the arrows starting at new vertices. Let C be an equivalence class of \sim in $\mathcal{T}_{\text{dc}}^p(n)$. The Σ_n -action on $\mathcal{T}_{\text{dc}}^p(n)$ restricts to a Σ_n -action on C . We have to show that the part of the map in Proposition 6 given as the appropriate quotient of

$$\coprod_{T \in C} \left(\bigotimes_{v \in V_{\text{old}}(T) \setminus U_{\text{old}}(T)} \mathcal{O}(\text{val}(v)) \right) \otimes e^{\square(U_{\text{old}}(T))} \square \coprod_{v \in V_{\text{new}}(T)} f(\text{val}(v)) \quad (*)$$

is a (trivial) cofibration in $\mathcal{C}[\Sigma_n]$. Let Γ be a doubly colored directed graph, where the arrows starting at new vertices are labelled, isomorphic to the objects of the same type underlying the objects from C . Set

$$\varphi := \left(\bigotimes_{v \in V_{\text{old}}(\Gamma) \setminus U_{\text{old}}(\Gamma)} \mathcal{O}(\text{val}(v)) \right) \otimes e^{\square(U_{\text{old}}(\Gamma))} \square \coprod_{v \in V_{\text{new}}(\Gamma)} f(\text{val}(v)) .$$

On φ there is an action of $\text{Aut}(\Gamma)$. Let t be the set of tails of Γ . There is an action of $\text{Aut}(\Gamma)$ on t . It is easily seen that the quotient of the map $(*)$ we are considering is isomorphic to $\varphi \times_{\text{Aut}(\Gamma)} \Sigma_t$. Hence we are finished if we show that φ is a (trivial) cofibration in $\mathcal{C}[\text{Aut}(\Gamma)]$. This is done by induction on the depth of Γ . Let $\Gamma_1, \dots, \Gamma_k$ be the different isomorphism types of doubly colored directed graphs, such that the arrows starting at new vertices are labelled, sitting at the initial vertex of Γ with multiplicities n_1, \dots, n_k and set $G_i := \text{Aut}(\Gamma_i)$, $i = 1, \dots, k$. Then, if the initial vertex of Γ is old, $\text{Aut}(\Gamma) = (\prod_{i=1}^k \Sigma_{n_i}) \ltimes (\prod_{i=1}^k G_i^{n_i})$, otherwise $\text{Aut}(\Gamma) = \prod_{i=1}^k G_i^{n_i}$, and the map φ is given like the map h in Lemma 4. Now the claim follows from Lemma 4 and the induction hypothesis. \square

Proof of Theorem 3. We apply Theorem 2 to the monad \mathbb{T} which maps X to $(FX)^\sharp$. It is known that $\text{Op}(\mathcal{C})$ is cocomplete. Since filtered colimits in $\text{Op}(\mathcal{C})$ are computed in $\mathcal{C}^{\mathbb{N}}$, it follows from Lemma 3 that those maps from FI -cell (resp. FJ -cell) whose domain is cofibrant in $\mathcal{C}^{\Sigma, \bullet}$ are cofibrations (resp. trivial cofibrations) in $\mathcal{C}^{\Sigma, \bullet}$.

It is clear that $\text{Op}(\mathcal{C})$ is right proper if \mathcal{C} is. If \mathcal{C} is left proper, then $\mathcal{C}^{\Sigma, \bullet}$ is left proper, and the pushout in $\text{Op}(\mathcal{C})$ by a cofibration whose domain is cofibrant in $\mathcal{C}^{\Sigma, \bullet}$ is a retract of a transfinite composition of pushouts by cofibrations in $\mathcal{C}^{\Sigma, \bullet}$, hence weak equivalences are preserved by these pushouts. \square

Remark 2. Let R be a commutative ring with unit and \mathcal{C} be the symmetric monoidal model category of unbounded chain complexes of R -modules with the projective model structure. Here the generating trivial cofibrations are all maps $0 \rightarrow D^n R$, $n \in \mathbb{Z}$. The $D^n R$ are clearly null-homotopic. From this it follows that for a generating trivial cofibration f in $\mathcal{C}^{\mathbb{N}}$ the codomain of the maps in Proposition 6 along which the pushouts are taken (these maps have domain 0, so the pushouts are trivial) are also null-homotopic by the homotopy which is on the summand corresponding to a tree $T \in \mathcal{T}_{\text{dc}}^p$ the sum over the homotopies from above over all new vertices of T (this homotopy factors through the quotient which is taken). Hence the conditions of Theorem 1 are fulfilled, so we get a model structure on $\text{Op}(\mathcal{C})$ which is the same as the one provided by [Hin1, Theorem 6.1.1].

Remark 3. One can use exactly the same methods as above to give the category of colored operads in \mathcal{C} for any set of labels the structure of a J -semi model category. In the case of unbounded complexes over a commutative unital ring as above this J -semi model structure is again a model structure.

4. ALGEBRAS

For an operad $\mathcal{O} \in \text{Op}(\mathcal{C})$ let us denote by $\text{Alg}(\mathcal{O})$ the category of algebras over \mathcal{O} . Let $F_{\mathcal{O}} : \mathcal{C} \rightarrow \text{Alg}(\mathcal{O})$ be the free algebra functor which is given by

$$F_{\mathcal{O}}(X) = \coprod_{n \geq 0} \mathcal{O}(n) \otimes_{\Sigma_n} X^{\otimes n}.$$

The right adjoint of $F_{\mathcal{O}}$ maps A to $A^{\#}$.

Remark 4. An \mathcal{O} -algebra structure on an object $A \in \mathcal{C}$ is the same as to give a map of operads $\mathcal{O} \rightarrow \text{End}^{\text{Op}}(A)$.

Lemma 5. Let I be a small category and let $D : I \rightarrow \text{Op}(\mathcal{C})$, $i \mapsto \mathcal{O}_i$, be a functor. Set $\mathcal{O} := \text{colim}_{i \in I} \mathcal{O}_i$ and let $A, B \in \mathcal{C}$. Then the following is valid.

1. To give an \mathcal{O} -algebra structure on A is the same as to give \mathcal{O}_i -algebra structures on A compatible with all transition maps in D .
2. Assume that A and B have \mathcal{O} -algebra structures and let $f : A \rightarrow B$ be a map in \mathcal{C} . Then f is a map of \mathcal{O} -algebras if and only if it is a map of \mathcal{O}_i -algebras for all $i \in D$.

Proof. The first part follows from the Remark above.

Let f be compatible with all \mathcal{O}_i -algebra structures. Then it can be checked directly that f is also compatible with the algebra structure on $\mathcal{O}' := \coprod_{i \in D} \mathcal{O}_i$. But since the maps $\mathcal{O}'(n) \rightarrow \mathcal{O}(n)$ are coequalizers in \mathcal{C} the claim follows. \square

The first main result of this section is

Theorem 4. *Let $\mathcal{O} \in \text{Op}(\mathcal{C})$ be cofibrant. Then the category $\text{Alg}(\mathcal{O})$ is a cofibrantly generated J -semi model category over \mathcal{C} with generating cofibrations $F_{\mathcal{O}}I$ and generating trivial cofibrations $F_{\mathcal{O}}J$. If \mathcal{C} is left proper (resp. right proper), then $\text{Alg}(\mathcal{O})$ is left proper relative to \mathcal{C} (resp. right proper). If the monoid axiom holds in \mathcal{C} , then $\text{Alg}(\mathcal{O})$ is a cofibrantly generated model category.*

We want to describe pushouts by free algebra maps. The following definition has its origin in [Hin2]Definitions 3.3.1 and 3.3.2].

Definition 4. 1. A doubly colored am-tree is the same as a doubly colored n -tree except that instead of the labeling of the tails every tail is marked by either a or m .
 2. A proper doubly colored am-tree is a doubly colored am-tree such that every arrow starting from an old vertex is either a tail or goes to a new vertex and every vertex with only tails as outgoing arrows is new and at least one of the outgoing tails is marked by m .

Note that in particular a proper doubly colored am-tree has no vertices of valency 0.

Let \mathcal{T}_{am} be the set of isomorphism classes of doubly colored am-trees and $\mathcal{T}_{\text{am}}^p$ the set of isomorphism classes of proper doubly colored am-trees. For $T \in \mathcal{T}_{\text{am}}$ let $a(T)$ be the set of tails of T marked by a and $m(T)$ the set of tails of T marked by m .

Let $T \in \mathcal{T}_{\text{am}}^p$. Similarly as in the case of operads there is the operation of changing a new vertex v of T into an old vertex and also of changing a tail marked by m into a tail marked by a . Denote the resulting trees in $\mathcal{T}_{\text{am}}^p$ by $\text{ch}_v(T)$ for $v \in V_{\text{new}}(T)$ and by $\text{ch}_t(T)$ for $t \in a(T)$. For $\mathcal{O} \in \text{Op}(\mathcal{C})$ and $A \in \text{Alg}(\mathcal{O})$ there is as in the operad case a concatenation map

$$\begin{aligned} \text{conc}_T^{\mathcal{O}}(v) : \mathcal{O}(\text{val}(v)) \otimes \bigotimes_{v' \in V_{\text{old}}(T)} \mathcal{O}(\text{val}(v')) \otimes A^{\otimes(a(T))} \longrightarrow \\ \bigotimes_{v' \in V_{\text{old}}(\text{ch}_T(v))} \mathcal{O}(\text{val}(v')) \otimes A^{\otimes(a(\text{ch}_T(v)))} \end{aligned}$$

induced by the operad maps of \mathcal{O} and the structure maps of A . There is also a concatenation map

$$\text{conc}_t^{\mathcal{O}, A}(T) : A \otimes \bigotimes_{v \in V(T)} \mathcal{O}(\text{val}(v)) \otimes A^{\otimes(a(T))} \longrightarrow \bigotimes_{v \in V(\text{ch}_T(t))} \mathcal{O}(\text{val}(v)) \otimes A^{\otimes(a(\text{ch}_T(t)))}$$

induced by the structure maps of the algebra A .

Proposition 7. *Let $\mathcal{O} \in \text{Op}(\mathcal{C})$ and $f : X \rightarrow Y$ and $\varphi : X \rightarrow \mathcal{O}^{\sharp}$ be maps in $\mathcal{C}^{\mathbb{N}}$. Let \mathcal{O}' be the pushout of \mathcal{O} by Ff with attaching map the adjoint of φ . Let A be an \mathcal{O}' -algebra and let $g : M \rightarrow N$ and $\psi : M \rightarrow A^{\sharp}$ be maps in \mathcal{C} . Let B be the pushout of A as \mathcal{O} -algebra by $F_{\mathcal{O}}(g)$ with attaching map the adjoint of ψ and B' the pushout of A as \mathcal{O}' -algebra by $F_{\mathcal{O}'}(g)$. Then the canonical map $h : B \rightarrow B'$ is given by a $\omega \times \omega \times (\omega + 1)$ -sequence $B' = \text{colim}_{(i,j,k)} B_{(i,j,k)}$, where for (i, j, k) a successor $B_{(i,j,k)}$ is a pushout of $B_{(i,j,k)-1}$ by the quotient of the map*

$$\coprod \left(\bigotimes_{v \in V_{\text{old}}(T) \setminus U_{\text{old}}(T)} \mathcal{O}(\text{val}(v)) \right) \otimes A^{\otimes(a(T))} \otimes e^{\square(U_{\text{old}}(T))} \square g^{\square(m(T))} \square \prod_{v \in V_{\text{new}}(T)} f(\text{val}(v)),$$

where the coproduct is over all $T \in \mathcal{T}_{\text{am}}^p$ with $\sharp V_{\text{new}}(T) = i$, $\sharp m(T) = j$ and $u_{\text{old}}(T) = k$, with respect to the equivalence relation which identifies for every isomorphism of directed graphs $\varphi : T \rightarrow T'$, $T, T' \in \mathcal{T}_{\text{am}}^p$, which respects the labeling of the tails and of the arrows which start at new vertices, the summands corresponding to T and T' by a map which is described on the \otimes -part of the summands involving vertices from $V_{\text{old}}(T) \setminus U_{\text{old}}(T)$ as in Proposition 5.3 and on the other parts by the identification of the indexing sets via φ . The attaching map is induced on the different parts of the domain of the above map by either the operation of removing a vertex of valency one, by changing a new vertex into an old vertex or by changing a tail labelled by m into a tail labelled by a and then by applying either a unit map, a map $\text{conc}_T(v)$ or a map $\text{conc}_T(t)$.

Proof. We have to do the same steps as in the proof of Proposition 6. Let C be the colimit described in the Proposition. The attaching maps are again well-defined because the various concatenation processes commute with each other and because of the symmetry properties of \mathcal{O} and the equivalence relations appearing in previous steps.

We equip C with an \mathcal{O}' -algebra structure: Let us define the structure map $\mathcal{O}'(n) \otimes A^{\otimes n} \rightarrow A$. For $T \in \mathcal{T}_{\text{dc}}^p(n)$ let $S(T)$ be as in the proof of Proposition 6. For $T \in \mathcal{T}_{\text{am}}^p$ let

$$S^a(T) := \left(\bigotimes_{v \in V_{\text{old}}(T)} \mathcal{O}(\text{val}(v)) \right) \otimes A^{\otimes(a(T))} \otimes N^{\otimes(m(T))} \otimes \bigotimes_{v \in V_{\text{new}}(T)} Y(\text{val}(v)).$$

Let $T \in \mathcal{T}_{\text{dc}}^p(n)$ and $T_i \in \mathcal{T}_{\text{am}}^p$, $i = 1, \dots, n$. We obtain a tree $\tilde{T} \in \mathcal{T}_{\text{am}}^p$ by glueing T_i to the tail of T labelled by i and then concatenating. By applying operad and algebra structure maps we get a map $S(T) \otimes S^a(T_1) \otimes \dots \otimes S^a(T_n) \rightarrow S^a(\tilde{T})$. It is then possible by similar considerations as in the proof of Proposition 6 to get from these maps the desired structure map of C . It is easy to see that these structure maps are associative and symmetric. Hence C is an \mathcal{O}' -algebra which receives an \mathcal{O} -algebra map from B and \mathcal{O}' -algebra maps from A and $F_{\mathcal{O}'}(N)$ which are compatible with each other in the obvious way.

We have to check that for an \mathcal{O}' -algebra D a map $c : C \rightarrow D$ is the same as a map of \mathcal{O}' -algebras $a : A \rightarrow D$ and a map $n : N \rightarrow A^\sharp$ which are compatible with each other. We get the maps a and n from c by the obvious compositions. Given a and n we first obtain a map of \mathcal{O} -algebras $B \rightarrow D$. Moreover for any $T \in \mathcal{T}_{\text{am}}^p$ there is a map $S^a(T) \rightarrow D$ by applying the \mathcal{O}' -algebra structure maps of D . It is then easy to check that these maps glue together to give the map c . These processes are inverses to each other. \square

Lemma 6. *Let the notation be as in the Proposition above. If \mathcal{O} is cofibrant as an object in $\mathcal{C}^{\Sigma, \bullet}$, A is cofibrant as an object in \mathcal{C} , f is a cofibration in $\mathcal{C}^{\mathbb{N}}$ and g is a cofibration in \mathcal{C} then the map $h : B \rightarrow B'$ is a cofibration in \mathcal{C} . If f or g is a trivial cofibration then so is h . If f or g is a trivial cofibration and A is arbitrary, then h lies in $(\mathcal{C} \otimes J)\text{-cof}$, hence is a weak equivalence if the monoid axiom holds in \mathcal{C} .*

Proof. Let \sim be the equivalence relation on $\mathcal{T}_{\text{am}}^p$ which identifies T and T' in $\mathcal{T}_{\text{am}}^p$ if there is an isomorphism of directed graphs $T \rightarrow T'$ which respects the labeling of

the tails and of the arrows starting at new vertices. Let C be an equivalence class of \sim in $\mathcal{T}_{\text{am}}^p$. We have to show that the appropriate quotient of the map

$$\coprod_{T \in C} \left(\bigotimes_{v \in V_{\text{old}}(T) \setminus U_{\text{old}}(T)} \mathcal{O}(\text{val}(v)) \right) \otimes A^{\otimes(a(T))} \otimes e^{\square(U_{\text{old}}(T))} \square g^{\square(m(T))} \square \coprod_{v \in V_{\text{new}}(T)} f(\text{val}(v))$$

is a (trivial) cofibration in \mathcal{C} (or lies in $(\mathcal{C} \otimes J)$ -cof under the assumptions of the last statement). This is done as in the proof of Lemma 3 by induction on the depth of the trees in C . This time instead of using Lemma 4 it is sufficient to use Lemma 2 applied to rings of the form $\mathbb{1}[\prod_{i=1}^k \Sigma_{n_i}]$. \square

Proof of Theorem 4. We apply Theorem 2 to the monad $\mathbb{T}_{\mathcal{O}}$ which maps X to $(F_{\mathcal{O}}X)^{\sharp}$. It is known that $\text{Alg}(\mathcal{O})$ is cocomplete. Since filtered colimits in $\text{Alg}(\mathcal{O})$ are computed in \mathcal{C} we are reduced to show that the pushout of an \mathcal{O} -algebra A which is cofibrant as an object in \mathcal{C} by a map in $F_{\mathcal{O}}I$ (resp. in $F_{\mathcal{O}}J$) is a cofibration (resp. trivial cofibration) in \mathcal{C} . Since \mathcal{O} is a retract of a cell operad (i.e. a cell complex in $\text{Op}(\mathcal{C})$) such a pushout is a retract of a pushout of the same kind with the additional hypothesis that \mathcal{O} is a cell operad. So let \mathcal{O} be a cell operad. Then the pushout in question is a transfinite composition of maps h as in Proposition 7, hence by Lemma 6 it is a (trivial) cofibration.

It is clear that $\text{Alg}(\mathcal{O})$ is right proper if \mathcal{C} is. The pushout in $\text{Alg}(\mathcal{O})$ by a cofibration whose domain is cofibrant in \mathcal{C} is a retract of a transfinite composition of pushouts by cofibrations in \mathcal{C} , hence if \mathcal{C} is left proper weak equivalences are preserved by these pushouts, so $\text{Alg}(\mathcal{C})$ is also left proper.

The last statement follows again from Lemma 6. \square

The second result concerning algebras is

Theorem 5. *Let \mathcal{O} be an operad in \mathcal{C} which is cofibrant as an object in \mathcal{C}^{Σ} . Then $\text{Alg}(\mathcal{O})$ is a cofibrantly generated J -semi model category with generating cofibrations $F_{\mathcal{O}}I$ and generating trivial cofibrations $F_{\mathcal{O}}J$. If \mathcal{C} is right proper, so is $\text{Alg}(\mathcal{O})$.*

The next result enables one to control pushouts of cofibrant algebras by free algebra maps.

For an ordinal λ denote by S_{λ} the set of all maps $f : \lambda \rightarrow \frac{1}{2}\mathbb{N}$ such that $f(i) \neq 0$ only for finitely many $i < \lambda$, if $f(i) \notin \mathbb{N}$ then $i > 0$ and $f(i') = 0$ for all $i' < i$ and if λ is a successor then $f(\lambda - 1) = 0$. For $f, f' \in S_{\lambda}$ say that $f < f'$ if there is an $i < \lambda$ such that $f(i') = f'(i')$ for all $i' > i$ and $f(i) < f'(i)$. With this ordering S_{λ} is well-ordered. For $i < \lambda$ denote by f_i the element of S_{λ} with $f_i(i) = \frac{1}{2}$ and $f_i(i') = 0$ for $i' \neq i$. Set $S_{\lambda,+} := S_{\lambda} \sqcup \{*\}$, where $*$ is by definition smaller than any other element in $S_{\lambda,+}$. Note that $f \in S_{\lambda,+}$ is a successor if and only if $f \neq *$ and $f(\lambda) \in \mathbb{N}$. For $f \in S_{\lambda,+}$ a successor let $|f| := \sum_{i < \lambda} f(i) \in \mathbb{N}$ and $\Sigma_f := \prod_{i < \lambda} \Sigma_{f(i)}$.

Proposition 8. *Let $\mathcal{O} \in \text{Op}(\mathcal{C})$ and $A = \text{colim}_{i < \lambda} A_i$ be a $F_{\mathcal{O}}(\text{Mor}(\mathcal{C}))$ -cell \mathcal{O} -algebra ($\text{Mor}(\mathcal{C})$ is the class of all morphisms in \mathcal{C}) with $A_0 = \mathcal{O}(0)$, where the transition maps $A_i \rightarrow A_{i+1}$ are pushouts of free \mathcal{O} -algebra maps on maps $g_i : K_i \rightarrow L_i$ in \mathcal{C} by maps adjoint to $\varphi_i : K_i \rightarrow A_i^{\sharp}$. Then A is a transfinite composition $A = \text{colim}_{f \in S_{\lambda,+}} A_f$ in \mathcal{C} such that*

1. $A_* = 0$ and $A_{f_i} = A_i$ for $i < \lambda$,

2. for $f \in S_\lambda$ such that for an $i_0 < \lambda$ we have $f(i_0) \notin \mathbb{N}$, there is for all $m \in \mathbb{N}$, successors $l \in S_{\lambda,+}$ with $l < f$ and $n := m + |l|$ a map

$$\Psi_{f,m,l} : \mathcal{O}(n) \otimes_{(\Sigma_m \times \Sigma_l)} \left(A_{i_0}^{\otimes m} \otimes \bigotimes_{i < \lambda} L_i^{\otimes l(i)} \right) \rightarrow A_f$$

compatible with the structure map $\mathcal{O}(n) \otimes_{\Sigma_n} A^{\otimes n} \rightarrow A$. By applying permutations to $\mathcal{O}(n)$ and the big bracket there are similar maps for other orders of the factors in the big bracket. These maps satisfy the following conditions:

- (a) They are compatible with the maps $L_i \rightarrow A_{i_0}$ for $i < i_0$. Moreover, if we replace a factor L_{i_0} by K_{i_0} we can either go to L_{i_0} or to A_{i_0} and apply suitable maps Ψ . Then the two compositions coincide.
- (b) They are associative in the following sense: Let $f_1, \dots, f_k \in S_\lambda$ be limit elements with $f_i < f$, $i = 1, \dots, k$, and let for each f_i be given m_i , l_i and n_i satisfying the same conditions as m , l and n for f . Let D_i be the domain of Ψ_{f_i, m_i, l_i} . Then the two possible ways to get from

$$\mathcal{O}(n) \otimes \left(\bigotimes_{i=1}^k D_i \right) \otimes A_{i_0}^{\otimes m} \otimes \bigotimes_{i < \lambda} L_i^{\otimes l(i)}$$

to A_f given by either applying the Ψ_{f_i, m_i, l_i} and then $\Psi_{f, m+k, l}$ or by applying the obvious operad structure maps and a suitable permutation of $\Psi_{f, m+\sum_{i=1}^k m_i, l+\sum_{i=1}^k l_i}$ coincide.

3. For any successor $f \in S_{\lambda,+}$ the map $A_{f-1} \rightarrow A_f$ is a pushout by

$$\mathcal{O}(|f|) \otimes_{\Sigma_f} \square_{i < \lambda} g_i^{\square f(i)},$$

where the attaching maps on the various parts of the domain of this map are induced from the maps in (2) (see below).

Proof. The whole Proposition is shown by induction on λ , so suppose that it is true for ordinals less than λ . We construct the map in 2, prove its properties and define the attaching map in 3 by transfinite induction: Suppose $f \in S_{\lambda,+}$ is a successor, that $A_{f'}$ is defined for $f' < f$ and that the map in 2 is defined for all limit elements $\tilde{f} \in S_\lambda$ with $\tilde{f} < f$. Let $i_0 \in \lambda$ with $f(i_0) > 0$ and let f' coincide with f except that $f'(i_0) = f(i_0) - 1$. The attaching map on the summand

$$S := \mathcal{O}(|f|) \otimes_{\Sigma_{f'}} \left(\left(\bigotimes_{i < i_0} L_i^{\otimes f(i)} \right) \otimes K_{i_0} \otimes L_{i_0}^{\otimes (f(i_0)-1)} \otimes \bigotimes_{i_0 < i < \lambda} L_i^{\otimes f(i)} \right)$$

of the domain of

$$\mathcal{O}(|f|) \otimes_{\Sigma_f} \square_{i < \lambda} g_i^{\square f(i)}$$

is given as follows: Let $\tilde{f}, l \in S_\lambda$ be defined by $\tilde{f}(i_0) = f(i_0) - \frac{1}{2}$, $l(i_0) = f(i_0) - 1$, $\tilde{f}(i) = l(i) = 0$ for $i < i_0$ and $\tilde{f}(i) = l(i) = f(i)$ for $i > i_0$. Let $m := 1 + \sum_{i < i_0} f(i)$. There is a canonical map

$$S \rightarrow \mathcal{O}(|f|) \otimes_{\Sigma_{f'}} \left(A_{i_0}^{\otimes (m-1)} \otimes A_{i_0} \otimes L_{i_0}^{\otimes (f(i_0)-1)} \otimes \bigotimes_{i_0 < i < \lambda} L_i^{\otimes f(i)} \right)$$

whose codomain maps naturally to the domain of $\Psi_{\tilde{f}, m, l}$. So we get maps $S \rightarrow A_{\tilde{f}} \rightarrow A_{f-1}$ the composition of which is the attaching map on the summand S .

These maps glue together for various summands S : There are two cases to distinguish. In the first one the intersection of two summands contains K_{i_0} twice. Then the two maps on this intersection coincide because of the symmetric group invariance. In the second case the intersection I contains $K_{i'_0}$ and K_{i_0} with $i'_0 < i_0$. Let \tilde{f} be as above and \tilde{f}' be similarly defined for i'_0 . Now the two properties 2(a) of the maps Ψ state that both maps $I \rightarrow A_f$ are equal the map induced by first mapping both $K_{i'_0}$ and K_{i_0} to A_{i_0} and then applying a suitable map Ψ .

Now suppose $f \in S_\lambda$ is a limit element with $f(i_0) \notin \mathbb{N}$ for some $i_0 < \lambda$. Define A_f as the colimit of the preceeding $A_{f'}$, $f' < f$. Let m, l and n be as in 2. We define $\Psi_{f,m,l}$ by induction on m and on S_{i_0} using the fact that $A_{i_0} = \text{colim}_{f' \in S_{i_0}} A_{f'}$ by induction hypothesis for the induction on λ . For abbreviation set $\mathcal{L} := \bigotimes_{i < \lambda} L_i^{\otimes l(i)}$. Let $f' \in S_{i_0}$ be a successor and let a map

$$\psi_{f'-1} : \mathcal{O}(n) \otimes \left(A_{i_0}^{\otimes(m-1)} \otimes A_{f'-1} \otimes \mathcal{L} \right) \rightarrow A_f$$

be already defined. $A_{f'}$ is a pushout of $A_{f'-1}$ by

$$\varphi : \mathcal{O}(|f'|) \otimes_{\Sigma_{f'}} \square_{i < i_0} \square_{f'(i)}.$$

Let $C := \bigotimes_{i < i_0} L_i^{\otimes f'(i)}$. Then the codomain of φ is $\mathcal{O}(|f'|) \otimes_{\Sigma_{f'}} C$. Moreover by induction hypothesis for the m -induction there is a map

$$\mathcal{O}(n + |f'| - 1) \otimes \left(A_{i_0}^{\otimes(m-1)} \otimes C \otimes \mathcal{L} \right) \rightarrow A_f,$$

hence by plugging in $\mathcal{O}(|f'|)$ into the m -th place of $\mathcal{O}(n)$ we get a map

$$\mathcal{O}(n) \otimes \left(A_{i_0}^{\otimes(m-1)} \otimes \mathcal{O}(|f'|) \otimes C \otimes \mathcal{L} \right) \rightarrow A_f.$$

This map and $\psi_{f'-1}$ glue together to a map $\psi_{f'}$: We have to show that they coincide after composition on domains of the form

$$\mathcal{O}(n) \otimes A_{i_0}^{\otimes(m-1)} \otimes \mathcal{O}(|f'|) \otimes_{\Sigma_{f''}} S' \otimes \mathcal{L}$$

for $\mathcal{O}(|f'|) \otimes_{\Sigma_{f''}} S'$ a summand of the domain of φ containing $K_{i'_0}$ for some $i'_0 < i_0$ (the definition of f'' is similar to the one of f'). To do this we can restrict for every A_{i_0} to objects $\mathcal{O}(|f'_i|) \otimes_{\Sigma_{f'_i}} C_i$, $i = 1, \dots, m-1$, for C_i of the same shape as C and $f'_i \in S_{i_0,+}$ successors. Then the two possible ways to get from

$$\mathcal{O}(n) \otimes \left(\bigotimes_{i=1}^{m-1} \mathcal{O}(|f'_i|) \otimes_{\Sigma_{f'_i}} C_i \right) \otimes \mathcal{O}(|f'|) \otimes_{\Sigma_{f''}} S' \otimes \mathcal{L}$$

to A_f can be compared by mapping $K_{i'_0}$ to $A_{i'_0}$, unwrapping the definitions of A_f and $\Psi_{f'-1}$ and using associativity of \mathcal{O} . We arrive at a map $\mathcal{O}(n) \otimes A_{i_0}^{\otimes m} \otimes \mathcal{L} \rightarrow A_f$. That it factors through the $(\Sigma_m \times \Sigma_l)$ -quotient follows after replacing $A_{i_0}^{\otimes m}$ by $\left(\bigoplus_{i=1}^k \mathcal{O}(|f'_i|) \otimes_{\Sigma_{f'_i}} C_i \right)^{\otimes m}$ (the C_i and f'_i as above) in the domain of this map since then the $(\Sigma_m \times \Sigma_l)$ -relation is obviously also valid in A_f .

Both properties 2(a) and (b) follow easily by the technique of restricting any appearing A_i by a factor $\mathcal{O}(|f'|) \otimes_{\Sigma_{f'}} C$.

Now using the maps Ψ and property 2(b) we can equip $\tilde{A} := \text{colim}_{f \in S_{\lambda,+}}$ with an \mathcal{O} -algebra structure (to do this accurately we have to enlarge λ a bit and the corresponding sequence by trivial pushouts).

We are left to prove the universal property for \tilde{A} by transfinite induction on λ . So let it be true for ordinals less than λ . If λ is a limit ordinal or the successor of a limit ordinal there is nothing to show. Let $\lambda = \alpha + 2$, let B be an \mathcal{O} -algebra and $A_\alpha \rightarrow B$ a map in $\text{Alg}(\mathcal{O})$ and $L_\alpha \rightarrow B^\sharp$ a map in \mathcal{C} such that these two maps are compatible via the attaching map. We define maps $A_f \rightarrow B$ by transfinite induction on $S_{\lambda,+}$, starting with the given map on $A_{f_\alpha} = A_\alpha$. So let $f_\alpha < f < \lambda$ be a successor. Since for any $i \leq \alpha$ there is a map $L_i \rightarrow B^\sharp$ we have a natural map

$$\mathcal{O}(|f|) \otimes_{\Sigma_f} \bigotimes_{i < \lambda} L_i^{\otimes f(i)} \rightarrow B$$

using the algebra structure maps of B . We have to show that this is compatible via the attaching map from the domain D of $\mathcal{O}(|f|) \otimes_{\Sigma_f} \prod_{i < \lambda} g_i^{\square f(i)}$ to A_{f-1} with the map $A_{f-1} \rightarrow B$ coming from the induction hypothesis. We check this again on a summand S of D containing some K_{i_0} . The attaching map on S is induced from $\Psi_{\tilde{f},m,l}$ as above. The canonical map from the domain of $\Psi_{\tilde{f},m,l}$ to B is compatible with $A_{\tilde{f}} \rightarrow B$ (as one checks again by replacing any A_{i_0} by essentially products of L_i 's as above), which together with the fact that $L_{i_0} \rightarrow B$ and $A_{i_0+1} \rightarrow B$ coincide on K_{i_0} implies the compatibility. By construction and the definition of the algebra structure on $A_{\alpha+1}$ the map $A_{\alpha+1} \rightarrow B$ just defined is an \mathcal{O} -algebra map.

If we have on the other hand a map of \mathcal{O} -algebras $A_{\alpha+1} \rightarrow B$ we can restrict it to get compatible maps $A_\alpha \rightarrow B$ and $L_\alpha \rightarrow B^\sharp$. These two assignments are inverse to each other. \square

Proof of Theorem 5. Let $\mathcal{O} \in \text{Op}(\mathcal{C})$ be cofibrant in \mathcal{C}^Σ . We have to show that the pushout of an \mathcal{O} -algebra such that the map from the initial \mathcal{O} -algebra to A is in $F_{\mathcal{O}}I$ -cof by a map from $F_{\mathcal{O}}I$ (resp. $F_{\mathcal{O}}J$) is a cofibration (resp. trivial cofibration) in \mathcal{C} . We can assume that A is a $F_{\mathcal{O}}I$ -cell \mathcal{O} -algebra, since in the general situation all maps we look at are retracts of corresponding maps in this situation. But if A is a cell \mathcal{O} -algebra our claim immediately follows from Proposition 8 and Lemma 2. \square

5. MODULE STRUCTURES

In this section we want to show that if \mathcal{C} is simplicial $\text{Alg}(\mathcal{O})$ is also a simplicial J -semi model category in the cases when the assumptions of Proposition 4 or Proposition 5 are fulfilled. Also $\text{Op}(\mathcal{C})$ is simplicial if \mathcal{C} is.

Definition 5. Let \mathcal{D} and \mathcal{E} be J -semi model categories (maybe over \mathcal{C}) and let \mathcal{S} be a model category. Then a Quillen bifunctor $\mathcal{D} \times \mathcal{S} \rightarrow \mathcal{E}$ is an adjunction of two variables $\mathcal{D} \times \mathcal{S} \rightarrow \mathcal{E}$ such that for any cofibration $g : K \rightarrow L$ in \mathcal{S} and fibration $p : Y \rightarrow Z$ in \mathcal{E} , the induced map

$$\text{Hom}_{r,\square}(g, p) : \text{Hom}_r(L, Y) \rightarrow \text{Hom}_r(L, Z) \times_{\text{Hom}_r(K, Z)} \text{Hom}_r(K, Y)$$

is a fibration in \mathcal{D} which is trivial if g or p is.

(See also [Hov1][Lemma 4.2.2].)

It follows that for f a cofibration in \mathcal{D} and g a cofibration in \mathcal{S} both of which have cofibrant domains the pushout $f \square g$ is a cofibration in \mathcal{E} which is trivial if f or g is.

Definition 6. Let \mathcal{D} be a J -semi model category (maybe over \mathcal{C}) and let \mathcal{S} be a symmetric monoidal model category. Then a Quillen \mathcal{S} -module structure on \mathcal{D} is a \mathcal{S} -module structure on \mathcal{D} such that the action map $\otimes : \mathcal{D} \times \mathcal{S} \rightarrow \mathcal{D}$ is a Quillen bifunctor and the map $X \otimes (QS) \rightarrow X \otimes S \cong X$ is a weak equivalence for all cofibrant $X \in \mathcal{D}$, where $QS \rightarrow S$ is a cofibrant replacement.

If \mathcal{D} has a Quillen \mathcal{S} -module structure we say that \mathcal{D} is an \mathcal{S} -module.

Let now \mathcal{S} be a symmetric monoidal model category where the tensor product is the categorical product on \mathcal{S} , so let us denote this by \times (e.g. $\mathcal{S} = \mathbf{SSet}$). Let be given a symmetric monoidal left Quillen functor $\mathcal{S} \rightarrow \mathcal{C}$.

Proposition 9. Let the situation be as above and assume that either $\mathbb{1}$ is cofibrant in \mathcal{S} or that \mathcal{C} is left proper and the maps in I have cofibrant domains. Let \mathcal{O} be an operad in \mathcal{C} which is either cofibrant in $Op(\mathcal{C})$ or cofibrant as an object in \mathcal{C}^Σ . Then the J -semi model category (in the first case over \mathcal{C}) $\text{Alg}(\mathcal{O})$ is naturally an \mathcal{S} -module and the functor $\mathcal{C} \rightarrow \text{Alg}(\mathcal{C})$ is an \mathcal{S} -module homomorphism.

Proof. Let $A \in \mathcal{C}$ and $K \in \mathcal{S}$. We denote by A^K the homomorphism object $\underline{\text{Hom}}(K, A) \in \mathcal{C}$. There is a map of operads

$$\text{End}^{\text{Op}}(A) \rightarrow \text{End}^{\text{Op}}(A^K),$$

which is described on as follows: We give the maps

$$\underline{\text{Hom}}(A^{\otimes n}, A) \rightarrow \underline{\text{Hom}}((A^K)^{\otimes n}, A^K)$$

on T -valued points ($T \in \mathcal{C}$): A map $T \otimes A^{\otimes n} \rightarrow A$ is sent to the composition

$$T \otimes (A^K)^{\otimes n} \rightarrow T \otimes (A^{\otimes n})^{K^n} \rightarrow T \otimes (A^{\otimes n})^K \rightarrow A^K,$$

where the second map is induced by the diagonal $K \rightarrow K^n$.

Hence for objects $K \in \mathcal{S}$ and $A \in \text{Alg}(\mathcal{O})$ the object $(A^\sharp)^K$ has a natural structure of \mathcal{O} -algebra given by the composition $\mathcal{O} \rightarrow \text{End}^{\text{Op}}(A) \rightarrow \text{End}^{\text{Op}}(A^K)$. We denote this \mathcal{O} -algebra by A^K .

For a fixed $K \in \mathcal{S}$ the functor $\text{Alg}(\mathcal{O}) \rightarrow \text{Alg}(\mathcal{O}), A \mapsto A^K$, has a left adjoint $A \mapsto A \otimes K$, which is given for a free \mathcal{O} -algebra $F_{\mathcal{O}}(X)$, $X \in \mathcal{C}$, by $F_{\mathcal{O}}(X) \otimes K = F_{\mathcal{O}}(X \otimes K)$ and which is defined in general by the requirement that $_ \otimes K$ respects coequalizers (note that every \mathcal{O} -algebra is a coequalizer of a diagram where only free \mathcal{O} -algebras appear). So we have a functor $\text{Alg}(\mathcal{O}) \times \mathcal{S} \rightarrow \text{Alg}(\mathcal{O})$.

Let now $B \in \text{Alg}(\mathcal{O})$ be fixed. By a similar argument as above the functor $\mathcal{S}^{\text{op}} \rightarrow \text{Alg}(\mathcal{O}), K \mapsto B^K$, has a left adjoint $A \mapsto \underline{\text{Hom}}^{\mathcal{S}}(A, B)$, which sends a free \mathcal{O} -algebra $F_{\mathcal{O}}(X)$, $X \in \mathcal{C}$, to the image of $\underline{\text{Hom}}(X, B^\sharp)$ in \mathcal{S} .

One checks that the functor $\text{Alg}(\mathcal{O}) \times \mathcal{S} \rightarrow \text{Alg}(\mathcal{O})$ we constructed defines an action of \mathcal{S} on $\text{Alg}(\mathcal{O})$.

It remains to show that this functor is a Quillen bifunctor and that the unit property is fulfilled. So let $g : K \rightarrow L$ be a cofibration in \mathcal{S} and $p : Y \rightarrow Z$ a fibration in $\text{Alg}(\mathcal{O})$. We have to show that $\text{Hom}_{\square, r}(g, p)$ is a fibration in $\text{Alg}(\mathcal{O})$, i.e. lies in $F_{\mathcal{O}}J\text{-inj}$. By adjointness this means that p has the right lifting property with respect to the maps $(F_{\mathcal{O}}f) \square g = F_{\mathcal{O}}(f \square g)$ for all $f \in J$, which is by adjointness the case because $f \square g$ is a trivial cofibration. When p or f is trivial we want to show

that $\text{Hom}_{\square, r}(g, p)$ lies in $F_{\mathcal{O}}I\text{-inj}$, so p should have the right lifting property with respect to the maps $F_{\mathcal{O}}(f \square g)$ for all $f \in I$, which is again the case by adjointness.

If $\mathbb{1}$ is cofibrant in \mathcal{S} we are ready. In the other case the unit property follows by transfinite induction from the explicit description of algebra pushouts, and hence the structure of cell algebras, given in Proposition 7 and the structure of cell algebras given in Proposition 8. \square

In a similar manner one shows

Proposition 10. *Let the situation be as before Proposition 9 and assume that either $\mathbb{1}$ is cofibrant in \mathcal{S} or that \mathcal{C} is left proper and the maps in I have cofibrant domains. Then $\text{Op}(\mathcal{C})$ is naturally an \mathcal{S} -module and the functor $\mathcal{C} \rightarrow \text{Op}(\mathcal{C})$ is an \mathcal{S} -module homomorphism.*

6. MODULES

Let $\mathcal{O} \in \text{Op}(\mathcal{C})$ and $A \in \text{Alg}(\mathcal{O})$. We denote the category of A -modules by $(\mathcal{O}, A)\text{-Mod}$, or $A\text{-Mod}$ if no confusion is likely. Let $F_{(\mathcal{O}, A)} : \mathcal{C} \rightarrow A\text{-Mod}$ (or F_A for short) be the free A -module functor. It is given by $M \mapsto U_{\mathcal{O}}(A) \otimes M$, where $U_{\mathcal{O}}(A)$ is the universal enveloping algebra of the \mathcal{O} -algebra A . Recall that $\text{Ass}(\mathcal{C})$ denotes the category of associative unital algebras in \mathcal{C} , and let F_{Ass} be the free associative algebra functor $\mathcal{C} \rightarrow \text{Ass}(\mathcal{C})$.

The main result of this section is

Theorem 6. *Let $\mathcal{O} \in \text{Op}(\mathcal{C})$ and $A \in \text{Alg}(\mathcal{O})$. Let one of the following two conditions be satisfied:*

1. *\mathcal{O} is cofibrant as an object in \mathcal{C}^{Σ} and A is a cofibrant \mathcal{O} -algebra.*
2. *\mathcal{O} is cofibrant in $\text{Op}(\mathcal{C})$ and A is cofibrant as an object in \mathcal{C} .*

Then there is cofibrantly generated model structure on $A\text{-Mod}$ with generating cofibrations $F_A I$ and generating trivial cofibrations $F_A J$. There is a right \mathcal{C} -module structure on $A\text{-Mod}$.

This theorem will follow from the fact that in each of the two cases the enveloping algebra $U_{\mathcal{O}}(A)$ is cofibrant in \mathcal{C} , since $A\text{-Mod}$ is canonically equivalent to $U_{\mathcal{O}}(A)\text{-Mod}$.

Note that there is a canonical surjection from the tensor algebra to the universal enveloping algebra

$$T_{\mathcal{O}}(A) := \coprod_{n \in \mathbb{N}} \mathcal{O}(n+1) \otimes_{\Sigma_n} A^{\otimes n} \rightarrow U_{\mathcal{O}}(A).$$

Proposition 11. *Let $\mathcal{O} \in \text{Op}(\mathcal{C})$ and $f : X \rightarrow Y$ and $\varphi : X \rightarrow \mathcal{O}^{\sharp}$ be maps in $\mathcal{C}^{\mathbb{N}}$. Let \mathcal{O}' be the pushout of \mathcal{O} by f with attaching map the adjoint of φ . Let A be a \mathcal{O}' -algebra. Then $U_{\mathcal{O}'}(A)$ is a pushout of $U_{\mathcal{O}}(A)$ in $\text{Ass}(\mathcal{C})$ by the map $F_{\text{Ass}}(\coprod_{n \in \mathbb{N}} f(n) \otimes A^{\otimes(n-1)})$ with attaching map the adjoint to the composition $\coprod_{n \in \mathbb{N}} X(n+1) \otimes A^{\otimes n} \rightarrow \coprod_{n \in \mathbb{N}} \mathcal{O}(n+1) \otimes_{\Sigma_n} A^{\otimes n} \rightarrow U_{\mathcal{O}}(A)$.*

Proof. (Compare to [Hin1][6.8.1. Lemma.]) A (\mathcal{O}', A) -module structure on a (\mathcal{O}, A) -module M is given by maps $Y(n+1) \otimes A^{\otimes n} \otimes M \rightarrow M$ for $n \in \mathbb{N}$ such that the compositions with $f(n+1) \otimes A^{\otimes n} \otimes M$ equals the composition $X(n+1) \otimes A^{\otimes n} \otimes M \rightarrow \mathcal{O}(n+1) \otimes A^{\otimes n} \otimes M \rightarrow M$. The same statement is true for a module structure under the described pushout algebra on a $U_{\mathcal{O}}(A)$ -module. \square

Corollary 1. *Let $\mathcal{O} \in \text{Op}(\mathcal{C})$ be cofibrant and let A be an \mathcal{O} -algebra which is cofibrant as an object in \mathcal{C} . Then $U_{\mathcal{O}}(A)$ is cofibrant in $\text{Ass}(\mathcal{C})$, in particular is cofibrant as an object in \mathcal{C} .*

Hence the first part of Theorem 6 is proven.

Corollary 2. *Let \mathcal{C} be left proper, let $\mathcal{O} \in \text{Op}(\mathcal{C})$ be cofibrant and let $A \rightarrow A'$ be a weak equivalence between \mathcal{O} -algebras both of which are cofibrant as objects in \mathcal{C} . Then the map $U_{\mathcal{O}}(A) \rightarrow U_{\mathcal{O}}(A')$ is a weak equivalence.*

We have an analogous result to Proposition 8 for the enveloping algebra of a cell algebra.

Proposition 12. *Let $\mathcal{O} \in \text{Op}(\mathcal{C})$ and $A = \text{colim}_{i < \lambda} A_i$ be a $F_{\mathcal{O}}(\text{Mor}(\mathcal{C}))$ -cell \mathcal{O} -algebra with $A_0 = \mathcal{O}(0)$, where the transition maps $A_i \rightarrow A_{i+1}$ are pushouts of free \mathcal{O} -algebra maps on maps $g_i : K_i \rightarrow L_i$ in \mathcal{C} by maps adjoint to $\varphi_i : K_i \rightarrow A_i^{\#}$. Then $U := U_{\mathcal{O}}(A)$ is a transfinite composition $U = \text{colim}_{f \in S_{\lambda,+}} U_f$ in \mathcal{C} such that*

1. $U_* = 0$ and $U_{f_i} = U_{\mathcal{O}}(A_i)$ for $i < \lambda$,
2. for $f \in S_{\lambda}$ such that for an $i_0 < \lambda$ we have $f(i_0) \notin \mathbb{N}$, there is for all $m \in \mathbb{N}$, successors $l \in S_{\lambda,+}$ with $l < f$ and $n := m + |l|$ a map

$$\mathcal{O}(n+1) \otimes_{(\Sigma_m \times \Sigma_l)} \left(A_{i_0}^{\otimes m} \otimes \bigotimes_{i < \lambda} L_i^{\otimes l(i)} \right) \rightarrow U_f$$

compatible with the map $\mathcal{O}(n+1) \otimes_{\Sigma_n} A^{\otimes n} \rightarrow U$ and

3. for any successor $f \in S_{\lambda,+}$ the map $U_{f-1} \rightarrow U_f$ is a pushout by

$$\mathcal{O}(|f|+1) \otimes_{\Sigma_f} \square_{i < \lambda} g_i^{\square f(i)},$$

where the attaching maps on the various parts of the domain of this map are induced from the maps in (2).

Proof. This Proposition is proven in essentially the same way as Proposition 8 except that this time we have to define associative algebra structures on the U_{f_i} and to verify the universal property stating the equivalence of module categories. For the associative algebra structure one uses the same formulas as for the tensor algebra and checks that they are compatible with the attaching maps. For the universal property one uses the fact that an A -module M is given by maps

$$\mathcal{O}(|f|+1) \otimes_{\Sigma_f} \left(\bigotimes_{i < \lambda} L_i^{\otimes f(i)} \right) \otimes M \rightarrow M$$

which are compatible in various ways the explicit formulation of which we leave to the reader. \square

Corollary 3. *For \mathcal{O} an operad in \mathcal{C} which is cofibrant in \mathcal{C}^{Σ} and A a cofibrant \mathcal{O} -algebra the enveloping algebra $U_{\mathcal{O}}(A)$ is cofibrant as an object in \mathcal{C} .*

Hence also the second part of Theorem 6 is proven.

Corollary 4. *Let \mathcal{C} be left proper, let $f : \mathcal{O} \rightarrow \mathcal{O}'$ be a weak equivalence between operads in \mathcal{C} both of which are cofibrant as objects in \mathcal{C}^Σ and let A be a cofibrant \mathcal{O} -algebra. Let A' be the pushforward of A with respect to f . Then the induced maps $A \rightarrow A'$ and $U_{\mathcal{O}}(A) \rightarrow U_{\mathcal{O}'}(A')$ are weak equivalences.*

Definition 7. *Let \mathcal{C} be left proper and let $\mathbb{1}$ and the domains of the maps in I be cofibrant in \mathcal{C} .*

1. *For $\mathcal{O} \in \text{Op}(\mathcal{C})$ define the derived category of \mathcal{O} -algebras $D\text{Alg}(\mathcal{O})$ to be $\text{Ho Alg}(Q\mathcal{O})$, where $Q\mathcal{O} \rightarrow \mathcal{O}$ is a cofibrant replacement in $\text{Op}(\mathcal{C})$. Define the derived 2-category of \mathcal{O} -algebras $D^{\leq 2}\text{Alg}(\mathcal{O})$ to be $\text{Ho}^{\leq 2}\text{Alg}(Q\mathcal{O})$.*
2. *For $\mathcal{O} \in \text{Op}(\mathcal{C})$ and $A \in \text{Alg}(\mathcal{O})$ define the derived category of A -modules $D(A\text{-Mod})$ to be $\text{Ho}(QA\text{-Mod})$, where $QA \rightarrow A$ is a cofibrant replacement of A in $\text{Alg}(Q\mathcal{O})$ with $Q\mathcal{O} \rightarrow \mathcal{O}$ a cofibrant replacement in $\text{Op}(\mathcal{C})$.*

Note that these definitions do not depend (up to equivalence up to unique isomorphism or up to equivalence up to isomorphism, which is itself defined up to unique isomorphism in the case of $D^{\leq 2}\text{Alg}(\mathcal{O})$) on the choices by Corollary 4 and [Hov2, Theorem 2.4], that if $\mathcal{O} \in \text{Op}(\mathcal{C})$ is cofibrant in \mathcal{C}^Σ there is a canonical equivalence $D\text{Alg}(\mathcal{O}) \sim \text{Ho Alg}(\mathcal{O})$ and that for a cofibrant $\mathcal{O} \in \text{Op}(\mathcal{C})$ and $A \in \text{Alg}(\mathcal{O})$ which is cofibrant in \mathcal{C} there is a canonical equivalence $D(A\text{-Mod}) \sim \text{Ho}(A\text{-Mod})$.

7. FUNCTORIALITY

In this section let \mathcal{C} be left proper and let $\mathbb{1}$ and the domains of the maps in I be cofibrant in \mathcal{C} .

Proposition 13. 1. *There is a well defined 2-functor*

$$\text{Ho}^{\leq 2}\text{Op}(\mathcal{C}) \rightarrow \mathbf{Cat} ,$$

$$\mathcal{O} \mapsto D\text{Alg}(\mathcal{O})$$

such that for any cofibrant operad \mathcal{O} in \mathcal{C} there is a canonical equivalence $D\text{Alg}(\mathcal{O}) \sim \text{Ho Alg}(\mathcal{O})$ and every functor in the image of this 2-functor has a right adjoint.

2. *For $\mathcal{O} \in \text{Op}(\mathcal{C})$ there is a well defined 2-functor*

$$D^{\leq 2}\text{Alg}(\mathcal{O}) \rightarrow \mathbf{Cat} ,$$

$$A \mapsto D(A\text{-Mod})$$

such that for any cofibrant $A \in \text{Alg}(Q\mathcal{O})$ ($Q\mathcal{O} \rightarrow \mathcal{O}$ a cofibrant replacement) there is a canonical equivalence $D(A\text{-Mod}) \sim \text{Ho}(A\text{-Mod})$ and every functor in the image of this 2-functor has a right adjoint.

Remark 5. *The 2-functor in the second part of the Proposition should be well defined for an object $\mathcal{O} \in \text{Ho}^{\leq 3}\text{Op}(\mathcal{C})$ and should depend on \mathcal{O} functorially.*

Proof. We prove the first part of the Proposition, the second one is similar. Let $\mathcal{O}, \mathcal{O}' \in \text{Op}(\mathcal{C})_{cf}$, $f, g \in \text{Hom}(\mathcal{O}, \mathcal{O}')$ and φ a 2-morphism from f to g in $\text{Ho}^{\leq 2}\text{Op}(\mathcal{C})$. First of all it is clear that the pushforward functor $f_* : \text{Alg}(\mathcal{O}) \rightarrow \text{Alg}(\mathcal{O}')$ is a left Quillen functor between J -semi model categories by the definition of the J -semi model structures. We have to show that φ induces a natural isomorphism between f_* and g_* on the level of homotopy categories. So let \mathcal{O}^\bullet be a cosimplicial frame on \mathcal{O} . φ can be represented by a chain of 1-simplices in $\text{Hom}(\mathcal{O}^\bullet, \mathcal{O}')$, and a homotopy between two representing chains by a chain of 2-simplices. So we can assume that φ is a 1-simplex, i.e. $\varphi \in \text{Hom}(\mathcal{O}^1, \mathcal{O}')$. We have maps $\mathcal{O} \sqcup \mathcal{O} \xrightarrow{i_0 \sqcup i_1} \mathcal{O}^1 \xrightarrow{p} \mathcal{O}$, and $\text{Ho Alg}(\mathcal{O}^1) \rightarrow \text{Ho Alg}(\mathcal{O})$ is an equivalence. Hence for $A \in \text{Ho Alg}(\mathcal{O})$ there is a unique isomorphism $\varphi'(A) : i_{0*}(A) \rightarrow i_{1*}(A)$ with $p_*(\varphi'(A)) = \text{Id}$. Then the $\varphi(\varphi'(A))$ define a natural isomorphism between $(\varphi \circ i_0)_*$ and $(\varphi \circ i_1)_*$. Now if we have a homotopy $\Phi \in \text{Hom}(\mathcal{O}^2, \mathcal{O}')$, the three natural transformations which are defined by the three 1-simplices of Φ are compatible, since on a given object they are the images in $\text{Ho Alg}(\mathcal{O}')$ of three compatible isomorphisms between the three possible images of A in $\text{Ho Alg}(\mathcal{O}^2)$. \square

Let $f : \mathcal{O} \rightarrow \mathcal{O}'$ be a map of operads in \mathcal{C} and let $A \in D^{\leq 2}\text{Alg}(\mathcal{O})$. Then there is an adjunction

$$D(A\text{-Mod}) \rightleftarrows D(f_*A\text{-Mod}) .$$

It follows that for $B \in D^{\leq 2}\text{Alg}(\mathcal{O}')$ there is also an adjunction

$$D(f^*B\text{-Mod}) \rightleftarrows D(B\text{-Mod}) .$$

Of course for A and B as above and a map $f_*A \rightarrow B$ there is a similar adjunction.

Now let \mathcal{D} be a second left proper symmetric monoidal cofibrantly generated model category with suitable smallness assumptions on the domains of the generating cofibrations and trivial cofibrations (depending on which definition of J -semi model category one takes) and with a cofibrant unit. Let $L : \mathcal{C} \rightarrow \mathcal{D}$ be a symmetric monoidal left Quillen functor with right adjoint R . For objects $X, Y \in \mathcal{D}$ there is always a natural map

$$R(X) \otimes R(Y) \rightarrow R(X \otimes Y)$$

adjoint to the map

$$F(R(X) \otimes R(Y)) \cong FR(X) \otimes FR(Y) \rightarrow X \otimes Y$$

which respects the associativity and commutativity isomorphisms (so R is a *pseudo symmetric monoidal* functor). It follows that L can be lifted to preserve operad, algebra and module structures.

Hence there is induced a pair of adjoint functors

$$\text{Op}(\mathcal{C}) \xrightleftharpoons[R_{\text{Op}}]{L_{\text{Op}}} \text{Op}(\mathcal{D}) ,$$

which is a Quillen adjunction between J -semi model categories by the definition of the model structures.

For $\mathcal{O} \in \text{Op}(\mathcal{C})$ there is induced a pair of adjoint functors

$$\text{Alg}(\mathcal{O}) \xrightleftharpoons[R_{\mathcal{O}}]{L_{\mathcal{O}}} \text{Alg}(L_{\text{Op}}(\mathcal{O})) ,$$

which is a Quillen adjunction between J -semi model categories in the cases where \mathcal{O} is either cofibrant in $\text{Op}(\mathcal{C})$ or cofibrant as an object in \mathcal{C}^Σ .

So for $\mathcal{O} \in \text{Op}(\mathcal{C})$, $\mathcal{O}' \in \text{Op}(\mathcal{D})$ and $f : L_{\text{Op}}(\mathcal{O}) \rightarrow \mathcal{O}'$ a map there are induced adjunctions

$$D\text{Alg}(\mathcal{O}) \rightleftarrows D\text{Alg}(\mathcal{O}') \quad \text{and}$$

$$D^{\leq 2}\text{Alg}(\mathcal{O}) \xrightleftharpoons{\Psi} D^{\leq 2}\text{Alg}(\mathcal{O}') .$$

Now let $A \in D^{\leq 2}\text{Alg}(\mathcal{O})$, $B \in D^{\leq 2}\text{Alg}(\mathcal{O}')$ and $g : \Psi(A) \rightarrow B$ be a map. Then there is induced an adjunction

$$D(A\text{-Mod}) \rightleftarrows D(B\text{-Mod}) .$$

All the adjunctions are compatible (in an appropriate weak categorical sense) with compositions of the maps which induce these adjunctions.

8. E_∞ -ALGEBRAS

Let \mathcal{N} be the operad in \mathcal{C} whose algebras are just the commutative unital algebras in \mathcal{C} , i.e. $\mathcal{N}(n) = \mathbb{1}$ for $n \in \mathbb{N}$, and let \mathcal{P} be the operad whose algebras are objects in \mathcal{C} pointed by $\mathbb{1}$, i.e. $\mathcal{P}(n) = \mathbb{1}$ for $n = 0, 1$, $\mathcal{P}(n) = 0$ otherwise. There is an obvious map $\mathcal{P} \rightarrow \mathcal{N}$.

Definition 8. 1. An E_∞ -operad in \mathcal{C} is an operad \mathcal{O} in \mathcal{C} which is cofibrant as an object in \mathcal{C}^Σ together with a map $\mathcal{O} \rightarrow \mathcal{N}$ which is a weak equivalence.
 2. A pointed E_∞ -operad in \mathcal{C} is an E_∞ -operad \mathcal{O} in \mathcal{C} together with a map $\mathcal{P} \rightarrow \mathcal{O}$ such that the composition with the map $\mathcal{O} \rightarrow \mathcal{N}$ is the canonical map $\mathcal{P} \rightarrow \mathcal{N}$.
 3. A unital E_∞ -operad in \mathcal{C} is a pointed E_∞ -operad in \mathcal{C} such that the map $\mathcal{P}(0) \rightarrow \mathcal{O}(0)$ is an isomorphism (this is the same as an E_∞ -operad \mathcal{O} in \mathcal{C} such that the map $\mathcal{O}(0) \rightarrow \mathcal{N}(0)$ is an isomorphism).

The unit $\mathbb{1}$ is an \mathcal{N} -algebra, hence it is an algebra for any E_∞ -operad.

We first want to show that under suitable conditions unital E_∞ -operads always exist.

For $\mathcal{O} \in \text{Op}(\mathcal{C})$ let us denote by $\mathcal{O}_{\leq 1}$ the operad with $\mathcal{O}_{\leq 1}(0) = \mathcal{O}(0)$, $\mathcal{O}_{\leq 1}(1) = \mathcal{O}(1)$ and $\mathcal{O}_{\leq 1}(n) = 0$ for $n > 1$. There is a canonical map $\mathcal{O}_{\leq 1} \rightarrow \mathcal{O}$ in $\text{Op}(\mathcal{C})$. If \mathcal{O} is an E_∞ -operad there is also a map $\mathcal{O}_{\leq 1} \rightarrow \mathcal{P}$ in $\text{Op}(\mathcal{C})$, and we denote by $\tilde{\mathcal{O}}$ the pushout of \mathcal{O} with respect to this map.

Lemma 7. *Let \mathcal{O} be an E_∞ -operad which admits a pointing.*

1. *Then there is a canonical equivalence $\text{Alg}^u(\mathcal{O}) \sim \text{Alg}(\tilde{\mathcal{O}})$, in particular an \mathcal{O} -algebra is unital if and only if it comes from an $\tilde{\mathcal{O}}$ -algebra.*
2. *Assume that \mathcal{C} is left proper, that $\mathbb{1}$ is cofibrant in \mathcal{C} and that \mathcal{O} is cofibrant in $\text{Op}(\mathcal{C})$. Then $\tilde{\mathcal{O}}$ is a unital E_∞ -operad in \mathcal{C} .*

Proof. By Lemma 5(1) an $\tilde{\mathcal{O}}$ -algebra A is the same as an \mathcal{O} -algebra A together with a map $\mathbb{1} \rightarrow A$ such that the structure map $\mathcal{O}(0) \rightarrow A$ is the composition $\mathcal{O}(0) \rightarrow \mathbb{1} \rightarrow A$. Hence a unital \mathcal{O} -algebra comes from an $\tilde{\mathcal{O}}$ -algebra. On the other hand if A is an $\tilde{\mathcal{O}}$ -algebra we have to show that the induced pointing $\mathbb{1} \rightarrow A$ is a map of algebras. This follows easily from the fact that the map $\mathcal{O}(0)$ has a right inverse (a pointing of \mathcal{O}). For the first part of the Lemma it remains to prove that an \mathcal{O} -algebra morphism between $\tilde{\mathcal{O}}$ -algebras is in fact an $\tilde{\mathcal{O}}$ -algebra morphism, which follows from Lemma 5(2).

Consider the commutative square

$$\begin{array}{ccc} \mathcal{O}(0) & \longrightarrow & F_{\mathcal{O}}(\mathbb{1}) \\ \downarrow & & \downarrow \\ \mathbb{1} & \longrightarrow & F_{\tilde{\mathcal{O}}}(\mathbb{1}) \end{array}$$

of \mathcal{O} -algebras and let P be the pushout of the left upper triangle of the square. We want to show that the canonical map $P \rightarrow F_{\tilde{\mathcal{O}}}(\mathbb{1})$ is an isomorphism. By the first part of the Lemma P is an $\tilde{\mathcal{O}}$ -algebra. Now again by the first part of the Lemma it is easily seen that P has the same universal property as $\tilde{\mathcal{O}}$ -algebra as $F_{\tilde{\mathcal{O}}}(\mathbb{1})$.

So the above square is a pushout square in $\text{Alg}(\mathcal{O})$, and hence by left properness of $\text{Alg}(\mathcal{O})$ over \mathcal{C} (Theorem 4) the right vertical arrow is a weak equivalence. This implies that $\mathcal{O} \rightarrow \tilde{\mathcal{O}}$ is a weak equivalence. It remains to prove that $\tilde{\mathcal{O}}$ is cofibrant as object in \mathcal{C}^{Σ} , which follows from Corollary 5. \square

Let us call a vertex $v \in V(T)$ of a tree $T \in \mathcal{T}$ a *no-tail* vertex if one cannot reach a tail from v . Let us call T *0-special* if the only no-tail vertices of T are vertices of valency 0. A proper 0-special doubly colored tree is a doubly colored tree which is 0-special such that any vertex of valency 0 is old. Let $\tilde{\mathcal{T}}_{\text{dc}}^p(n)$ be the set of isomorphism classes of such trees with n tails.

Lemma 8. *Let $\mathcal{O} = \text{colim}_{i < \lambda} \mathcal{O}_i$ be an operad in \mathcal{C} such that the transition maps $\mathcal{O}_i \rightarrow \mathcal{O}_{i+1}$ are pushouts of free operad maps on maps $g_i : K_i \rightarrow L_i$ in $\mathcal{C}^{\mathbb{N}}$ and such that \mathcal{O}_0 is the initial operad. Let $E \in \mathcal{C}$ and let $\mathcal{O}(0) \rightarrow E$ be a morphism in \mathcal{C} . Let \tilde{E} be the operad with $\tilde{E}(0) = E$, $\tilde{E}(1) = \mathbb{1}$ and $\tilde{E}(n) = 0$ for $n > 1$. Let the squares*

$$\begin{array}{ccc} \mathcal{O}_{i, \leq 1} & \longrightarrow & \mathcal{O}_i \\ \downarrow & & \downarrow \\ \tilde{E} & \longrightarrow & \tilde{\mathcal{O}}_i \end{array}$$

be pushout squares in $\text{Op}(\mathcal{C})$, where either $i < \lambda$ or i is the blanket. Then $\tilde{\mathcal{O}} = \text{colim}_{i < \lambda} \tilde{\mathcal{O}}_i$, and every map $\tilde{\mathcal{O}}_i \rightarrow \tilde{\mathcal{O}}_{i+1}$ is a $\omega \times (\omega + 1)$ -sequence in $\mathcal{C}^{\mathbb{N}}$ as in Proposition 6, where for $j < \omega$ $\mathcal{O}_{(i,j)}(n)$ is a pushout of $\mathcal{O}_{(i,j)-1}(n)$ in \mathcal{C} by the quotient of the map

$$\coprod \left(\bigotimes_{v \in V_{\text{old}}(T) \setminus U_{\text{old}}(T)} \tilde{\mathcal{O}}_i(\text{val}(v)) \right) \otimes e^{\square(U_{\text{old}}(T))} \square_{v \in V_{\text{new}}(T)} \square g_i(\text{val}(v)),$$

where the coproduct is over all $T \in \tilde{T}_{\text{dc}}^p(n)$ with $\sharp V_{\text{new}}(T) = i$ and $u_{\text{old}}(T) = j$, with respect to an equivalence relation analogous to the one in Proposition 6. In particular we have $\tilde{\mathcal{O}}_i(0) = E$ for all $i < \lambda$ or i the blanket.

Corollary 5. *Let the notation be as in the Lemma above and assume that the maps g_i are cofibrations in $\mathcal{C}^{\mathbb{N}}$ and that E is cofibrant in \mathcal{C} . Then the operad $\tilde{\mathcal{O}}$ is cofibrant in $\mathcal{C}^{\Sigma, \bullet}$.*

Proof. The proof is along the same lines as the proof of Lemma 3. \square

For the rest of this section let us fix a pointed E_{∞} -operad \mathcal{O} in \mathcal{C} . An \mathcal{O} -algebra A is naturally pointed, i.e. there is a canonical map $\mathbb{1} \rightarrow A$, but note that this need not be a map of algebras. If it is, we say that A is a *unital* \mathcal{O} -algebra. Let us denote the category of unital \mathcal{O} -algebras by $\text{Alg}^u(\mathcal{O})$. This is just the category of objects in $\text{Alg}(\mathcal{O})$ under $\mathbb{1}$. If \mathcal{O} is unital, then every \mathcal{O} -algebra is unital.

Lemma 9. *If $\mathbb{1}$ is cofibrant in \mathcal{C} and \mathcal{O} is cofibrant in $\text{Op}(\mathcal{C})$ there is a J -semi model structure on $\text{Alg}^u(\mathcal{O})$ over \mathcal{C} .*

Proof. In any J -semi model category \mathcal{D} over \mathcal{C} the category of objects under an object from \mathcal{D} which becomes cofibrant in \mathcal{C} is again a J -semi model category over \mathcal{C} . \square

Lemma 10. *Assume that \mathcal{C} is left proper and that the domains of the maps in I are cofibrant. Let $A \in \text{Alg}(\mathcal{O})$ be cofibrant. Then the canonical map of A -modules $U_{\mathcal{O}}(A) \rightarrow A$ adjoint to the pointing $\mathbb{1} \rightarrow A$ is a weak equivalence.*

Proof. We can assume that A is a cell \mathcal{O} -algebra. It is easy to see that the map $U_{\mathcal{O}}(A) \rightarrow A$ is compatible with the descriptions of A and $U_{\mathcal{O}}(A)$ in Proposition 8 and Proposition 12 as transfinite compositions, and the map ψ from the map of part (3) of Proposition 12 to the map of part (3) of Proposition 8 is induced by the map $\mathcal{O}(|f| + 1) = \mathcal{O}(|f| + 1) \otimes \mathbb{1}^{\otimes |f|} \otimes \mathbb{1} \rightarrow \mathcal{O}(|f| + 1) \otimes \mathcal{O}(1)^{\otimes |f|} \otimes \mathcal{O}(0) \rightarrow \mathcal{O}(|f|)$, which itself is induced by the unit, the pointing and a structure map of \mathcal{O} . Since \mathcal{O} is an E_{∞} -operad this is a weak equivalence, hence since the domains of the maps in I are cofibrant ψ is a weak equivalence. Now the claim follows by transfinite induction and left properness of \mathcal{C} . \square

Corollary 6. *Assume that \mathcal{C} is left proper, that the domains of the maps in I are cofibrant and that \mathcal{O} is cofibrant in $\text{Op}(\mathcal{C})$. Let $A \in \text{Alg}(\mathcal{O})$ be cofibrant as object in \mathcal{C} . Then the canonical map of A -modules $U_{\mathcal{O}}(A) \rightarrow A$ adjoint to the pointing $\mathbb{1} \rightarrow A$ is a weak equivalence.*

Proof. Let $QA \rightarrow A$ be a cofibrant replacement. Then in the commutative square

$$\begin{array}{ccc} U_{\mathcal{O}}(QA) & \longrightarrow & U_{\mathcal{O}}(A) \\ \downarrow & & \downarrow \\ QA & \longrightarrow & A \end{array}$$

the horizontal maps are weak equivalences (the upper one by Corollary 2) and the left vertical arrow is a weak equivalence by the Lemma above, hence the right vertical map is also a weak equivalence. \square

Corollary 7. *Assume that \mathcal{C} is left proper and that the domains of the maps in I are cofibrant. Let $A \rightarrow A'$ be a weak equivalence between cofibrant \mathcal{O} -algebras. Then the map $U_{\mathcal{O}}(A) \rightarrow U_{\mathcal{O}}(A')$ is also a weak equivalence.*

Proof. This follows immediately from Lemma 10. \square

This Corollary has the consequence that under the assumptions of the Corollary there is a canonical equivalence $D(A\text{-Mod}) \sim \text{Ho } A\text{-Mod}$ for a cofibrant \mathcal{O} -algebra A .

9. \mathbb{S} -MODULES AND ALGEBRAS

In this section we generalize the theories developed in [EKMM] and [KM].

Definition 9. (I) *A symmetric monoidal category with pseudo-unit is a category \mathcal{D} together with*

- *a functor $-\boxtimes -: \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$,*
- *natural isomorphisms $(X \boxtimes Y) \boxtimes Z \rightarrow X \boxtimes (Y \boxtimes Z)$ and $X \boxtimes Y \rightarrow Y \boxtimes X$ which satisfy the usual equations and*
- *an object $\mathbb{1} \in \mathcal{D}$ with morphisms $\mathbb{1} \boxtimes X \rightarrow X$ (and hence morphisms $X \boxtimes \mathbb{1} \rightarrow X$ induced by the symmetry isomorphisms) such that the diagram*

$$\begin{array}{ccc} \mathbb{1} \boxtimes (X \boxtimes Y) & \xrightarrow{\quad} & X \boxtimes Y \\ \downarrow & \nearrow & \\ (\mathbb{1} \boxtimes X) \boxtimes Y & & \end{array}$$

commutes and such that the two possible maps $\mathbb{1} \boxtimes \mathbb{1} \rightarrow \mathbb{1}$ agree.

(II) *A symmetric monoidal functor between symmetric monoidal categories with pseudo-unit \mathcal{D} and \mathcal{D}' is a functor $F : \mathcal{D} \rightarrow \mathcal{D}'$ together with natural isomorphisms $F(X) \boxtimes F(Y) \rightarrow F(X \boxtimes Y)$ compatible with the associativity and commutativity isomorphisms and with a map $F(\mathbb{1}_{\mathcal{D}}) \rightarrow \mathbb{1}_{\mathcal{D}'}$ compatible with the unit maps.*

Definition 10. (I) *A symmetric monoidal model category with weak unit is a model category \mathcal{D} which is a symmetric monoidal category with pseudo-unit such that the functor $\mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ has the structure of a Quillen bifunctor ([Hov1, p. 108]) and such that the composition $Q\mathbb{1} \boxtimes X \rightarrow \mathbb{1} \boxtimes X \rightarrow X$ is a weak equivalence for all cofibrant $X \in \mathcal{D}$, where $Q\mathbb{1} \rightarrow \mathbb{1}$ is a cofibrant replacement.*

(II) *A symmetric monoidal Quillen functor between symmetric monoidal model categories with weak unit \mathcal{D} and \mathcal{D}' is a left Quillen functor $\mathcal{D} \rightarrow \mathcal{D}'$ which is a symmetric monoidal functor between symmetric monoidal categories with pseudo-unit such that the composition $F(Q\mathbb{1}_{\mathcal{D}}) \rightarrow F(\mathbb{1}_{\mathcal{D}}) \rightarrow \mathbb{1}_{\mathcal{D}'}$ is a weak equivalence.*

The homotopy category of a symmetric monoidal model category with weak unit is a closed symmetric monoidal category.

Let us assume now that \mathcal{C} is either simplicial (i.e. there is a symmetric monoidal left Quillen functor $\mathbf{SSet} \rightarrow \mathcal{C}$) or that there is a symmetric monoidal left Quillen functor $\text{Comp}_{\geq 0}(\mathbf{Ab}) \rightarrow \mathcal{C}$, where $\text{Comp}_{\geq 0}(\mathbf{Ab})$ is endowed with the projective model structure. In both cases we denote by \mathcal{L} the image of the linear isometries

operad in $\text{Op}(\mathcal{C})$ via either the simplicial complex functor or the simplicial complex functor followed by the normalized chain complex functor. Clearly \mathcal{L} is a unital E_∞ -operad. Let $\mathbb{S} := \mathcal{L}(1)$. \mathbb{S} is a ring with unit in \mathcal{C} which is cofibrant as an object in \mathcal{C} .

As in [EKMM] or [KM] we define a tensor product on $\mathbb{S}\text{-Mod}$ by

$$M \boxtimes N := \mathcal{L}(2) \otimes_{\mathbb{S} \otimes \mathbb{S}} M \otimes N.$$

[KM, Theorem V.1.5] and [KM, Lemma V.1.6] also work in our context, hence $\mathbb{S}\text{-Mod}$ is a symmetric monoidal category with pseudo-unit.

There is an internal Hom in $\mathbb{S}\text{-Mod}$ given by

$$\underline{\text{Hom}}^{\boxtimes}(M, N) := \underline{\text{Hom}}_{\mathbb{S}}(\mathcal{L}(2) \otimes_{\mathbb{S}} M, N),$$

where, when forming $\mathcal{L}(2) \otimes_{\mathbb{S}} M$, \mathbb{S} acts on $\mathcal{L}(2)$ through $\mathbb{S} = \mathbb{1} \otimes \mathbb{S} \rightarrow \mathbb{S} \otimes \mathbb{S}$, when forming $\underline{\text{Hom}}_{\mathbb{S}}$, \mathbb{S} acts on $\mathcal{L}(2) \otimes_{\mathbb{S}} M$ via its left action on $\mathcal{L}(2)$ and the left action of \mathbb{S} on $\underline{\text{Hom}}^{\boxtimes}(M, N)$ is induced through the right action of \mathbb{S} on $\mathcal{L}(2)$ through $\mathbb{S} = \mathbb{S} \otimes \mathbb{1} \rightarrow \mathbb{S} \otimes \mathbb{S}$.

There is an augmentation $\mathbb{S} \rightarrow \mathbb{1}$ which is a map of algebras with unit.

Proposition 14. *The category $\mathbb{S}\text{-Mod}$ is a cofibrantly generated symmetric monoidal model category with weak unit with generating cofibrations $\mathbb{S} \otimes I$ and generating trivial cofibrations $\mathbb{S} \otimes J$. The functor $\mathcal{C} \rightarrow \mathbb{S}\text{-Mod}$, $X \mapsto \mathbb{S} \otimes X$, is a Quillen equivalence, and its left inverse, the functor $\mathbb{S}\text{-Mod} \rightarrow \mathcal{C}$, $M \mapsto M \otimes_{\mathbb{S}} \mathbb{1}$, is a symmetric monoidal Quillen equivalence. Moreover there is a closed action of \mathcal{C} on $\mathbb{S}\text{-Mod}$.*

Proof. That $R\text{-Mod}$ is a cofibrantly generated model category together with a closed action of \mathcal{C} on it is true for any associative unital ring R in \mathcal{C} which is cofibrant as an object in \mathcal{C} .

Let f and g be cofibrations in \mathcal{C} . The \boxtimes -pushout product of $\mathbb{S} \otimes f$ and $\mathbb{S} \otimes g$ is isomorphic to $\mathcal{L}(2) \otimes (f \square g)$. As a left \mathbb{S} -module $\mathcal{L}(2)$ is (non canonically) isomorphic to \mathbb{S} , hence $\mathcal{L}(2) \otimes (f \square g)$ is a cofibration $\mathbb{S}\text{-Mod}$, and it is trivial if one of f or g is trivial. To show that for a cofibrant \mathbb{S} -module M the map $Q\mathbb{1} \boxtimes M \rightarrow M$ is a weak equivalence we can assume that M is a cell \mathbb{S} -module and we can take $Q\mathbb{1} = \mathbb{S}$. Then M is a transfinite composition where the transition maps are pushouts of maps $f : \mathbb{S} \otimes K \rightarrow \mathbb{S} \otimes L$, where $K \rightarrow L$ is a cofibration in \mathcal{C} with K cofibrant. But the composition $\mathbb{S} \boxtimes \mathbb{S} \rightarrow \mathbb{1} \boxtimes \mathbb{S} \rightarrow \mathbb{S}$ is a weak equivalence between cofibrant objects in $\mathbb{S}\text{-Mod}$, hence the composition $\mathbb{S} \boxtimes f \rightarrow \mathbb{1} \boxtimes f \rightarrow f$ is a weak equivalence between cofibrations in $\mathbb{S}\text{-Mod}$. So by transfinite induction the composition $\mathbb{S} \boxtimes M \rightarrow \mathbb{1} \boxtimes M \rightarrow M$ is a weak equivalence between cofibrant objects in $\mathbb{S}\text{-Mod}$. \square

Note that in the simplicial case $\mathbb{1} \boxtimes \mathbb{S}$ is cofibrant in \mathcal{C} , hence for cofibrant M both maps $\mathbb{S} \boxtimes M \rightarrow \mathbb{1} \boxtimes M \rightarrow M$ are weak equivalences.

Let $\mathbb{S}\text{-Mod}^u$ be the category of unital \mathbb{S} -modules, i.e. the objects in $\mathbb{S}\text{-Mod}$ under $\mathbb{1} \in \mathbb{S}\text{-Mod}$. For $M \in \mathbb{S}\text{-Mod}^u$ and $N \in \mathbb{S}\text{-Mod}$ there are the products $M \triangleleft N$ and $N \triangleright M$, and for $M, N \in \mathbb{S}\text{-Mod}^u$ there is the product $M \square N$. These products are defined as in [KM, Definition V.2.1] and [KM, Definition V.2.6].

$\mathbb{S}\text{-Mod}^u$ is a symmetric monoidal category with \square as tensor product.

Analogous to [KM, Theorem V.3.1] and [KM, Theorem V.3.3] we have

Proposition 15. • $\text{Alg}(\mathcal{L})$ is naturally equivalent to the category of commutative rings with unit in $\mathbb{S}\text{-Mod}^u$. Hence for $A, B \in \text{Alg}(\mathcal{L})$ there is a natural isomorphism $A \sqcup B \cong A \sqcap B$.

- For $A \in \text{Alg}(\mathcal{L})$ an A -module M is the same as an \mathbb{S} -module M together with a map $A \triangleleft M \rightarrow M$ satisfying the usual identities.

For $A \in \text{Alg}(\mathcal{L})$ let $\text{Comm}(A)$ be the category of commutative unital A -algebras in $\mathbb{S}\text{-Mod}^u$, i.e. the objects in $\text{Alg}(\mathcal{L})$ under A . In particular we have $\text{Alg}(\mathcal{L}) \sim \text{Comm}(\mathbb{1}_{\mathbb{S}}) =: \text{Comm}_{\mathcal{C}}$, where we denote by $\mathbb{1}_{\mathbb{S}}$ the algebra $\mathbb{1}$ in $\mathbb{S}\text{-Mod}^u$.

For the rest of the section let us make the following

Assumption 1. The model category \mathcal{C} is left proper and $\mathbb{1}$ and the domains of the maps in I are cofibrant in \mathcal{C} .

Corollary 8. • $\text{Comm}_{\mathcal{C}}$ is a cofibrantly generated J -semi model category.

- For any cofibrant $A \in \text{Comm}_{\mathcal{C}}$ the category $\text{Comm}(A)$ is also a cofibrantly generated J -semi model category.
- If $A \rightarrow B$ is a weak equivalence between cofibrant $A, B \in \text{Comm}_{\mathcal{C}}$, then the induced functor $\text{Comm}(A) \rightarrow \text{Comm}(B)$ is a Quillen equivalence.

Proof. Follows from Theorem 5. □

Definition 11. For $A \in \text{Comm}_{\mathcal{C}}$ let $D\text{Comm}(A)$ be $\text{Ho Comm}(QA)$ for $QA \rightarrow A$ a cofibrant replacement in $\text{Comm}_{\mathcal{C}}$, and let $D^{\leq 2}\text{Comm}(A) := \text{Ho}^{\leq 2}\text{Comm}(QA)$. The 2-functor $\text{Comm}_{\mathcal{C}} \rightarrow \mathbf{Cat}$, $A \mapsto D\text{Comm}(A)$, descends to a 2-functor $D^{\leq 2}\text{Comm}_{\mathcal{C}} \rightarrow \mathbf{Cat}$, $A \mapsto D\text{Comm}(A)$.

Let $A \in \text{Comm}_{\mathcal{C}}$ and $M, N \in A\text{-Mod}$. As in [KM, Definition V.5.1] or [KM, Remark V.5.2] we define the tensor product $M \boxtimes_A N$ as the coequalizer in the diagram

$$(M \triangleright A) \boxtimes N \cong M \boxtimes (A \triangleleft N) \xrightleftharpoons[\text{Id} \boxtimes m]{m \boxtimes \text{Id}} M \boxtimes N \longrightarrow M \boxtimes_A N \quad \text{or}$$

$$M \boxtimes A \boxtimes N \xrightleftharpoons[\text{Id} \boxtimes m]{m \boxtimes \text{Id}} M \boxtimes N \longrightarrow M \boxtimes_A N .$$

With this product the category $A\text{-Mod}$ has the structure of a symmetric monoidal category with pseudo-unit, where the pseudo-unit is A . As for \mathbb{S} -modules one can define products \triangleleft_A , \triangleright_A and \square_A . There is also an analogue of Proposition 15 for A -algebras and modules over A -algebras.

The free A -module functor $\mathbb{S}\text{-Mod} \rightarrow A\text{-Mod}$ is given by $M \mapsto A \triangleleft M$. More generally for $A \rightarrow B$ a map in $\text{Comm}_{\mathcal{C}}$ the pushforward of modules is given by $M \mapsto B \triangleleft_A M$. In particular there is a canonical isomorphism of A -modules $U_{\mathcal{L}}(A) \cong A \triangleleft \mathbb{S}$.

Lemma 11. Let $A \rightarrow B$ be a map in $\text{Comm}_{\mathcal{C}}$, let $M, N \in A\text{-Mod}$ and $P \in B\text{-Mod}$. Then there are canonical isomorphisms

$$M \boxtimes_A P \cong (B \triangleleft_A M) \boxtimes_B P \quad \text{and}$$

$$(B \triangleleft_A M) \boxtimes_B (B \triangleleft_A N) \cong B \triangleleft_A (M \boxtimes_A N) .$$

Proof. Similar to the proof of [KM, Proposition V.5.8]. \square

For $M, N \in A\text{-Mod}$ define the internal Hom $\underline{\text{Hom}}_A^\boxtimes(M, N)$ in $A\text{-Mod}$ as the equalizer

$$\underline{\text{Hom}}_A^\boxtimes(M, N) \longrightarrow \underline{\text{Hom}}^\boxtimes(M, N) \rightrightarrows \underline{\text{Hom}}^\boxtimes(A \triangleleft M, N)$$

like in [KM, Definition V.6.1].

Proposition 16. *For a cofibrant $A \in \text{Comm}_{\mathcal{C}}$ the category $A\text{-Mod}$ is a cofibrantly generated symmetric monoidal model category with weak unit with generating cofibrations $A \triangleleft (\mathbb{S} \otimes I)$ and generating trivial cofibrations $A \triangleleft (\mathbb{S} \otimes J)$. If $f : A \rightarrow B$ is a map in $\text{Comm}_{\mathcal{C}}$ between cofibrant algebras the pushforward $f_* : A\text{-Mod} \rightarrow B\text{-Mod}$ is a symmetric monoidal Quillen functor which is a Quillen equivalence if f is a weak equivalence.*

Proof. $A\text{-Mod}$ is a cofibrantly generated model category by Theorem 6(1). Let f and g be cofibrations in \mathcal{C} . By Lemma 11 the \boxtimes_A -pushout product of the maps $A \triangleleft (\mathbb{S} \otimes f)$ and $A \triangleleft (\mathbb{S} \otimes g)$ is given by $A \triangleleft (\mathcal{L}(2) \otimes (f \square g))$, hence since $\mathcal{L}(2) \cong \mathbb{S}$ as \mathbb{S} -modules this is a cofibration in $A\text{-Mod}$, and it is trivial if one of f or g is trivial.

Note that $A \triangleleft \mathbb{S}$ is cofibrant in $A\text{-Mod}$ and that the map $A \triangleleft \mathbb{S} \cong U_{\mathcal{C}}(A) \rightarrow A$ is a weak equivalence by Lemma 10. So we have to show that for cofibrant $M \in A\text{-Mod}$ the map $(A \triangleleft \mathbb{S}) \boxtimes_A M \rightarrow M$ is a weak equivalence, which follows from the fact that the maps of the form $(A \triangleleft \mathbb{S}) \boxtimes_A (A \triangleleft (\mathbb{S} \otimes f)) \rightarrow A \triangleleft (\mathbb{S} \otimes f)$ for cofibrations $f \in \mathcal{C}$ with cofibrant domain are weak equivalences between cofibrations in $A\text{-Mod}$. The first part of the last statement follows from Lemmas 11 and 10, and the second part by Corollary 7. \square

For $A \in D^{\leq 2}\text{Comm}_{\mathcal{C}}$ there is unambiguously defined the closed symmetric monoidal category $D(A\text{-Mod})$ with tensor product denoted by \otimes_A . The assignment $A \mapsto D(A\text{-Mod})$ is a 2-functor $D^{\leq 2}\text{Comm}_{\mathcal{C}} \rightarrow \mathbf{Cat}^{\text{sm}}$, where \mathbf{Cat}^{sm} is the 2-category of symmetric monoidal categories, such that the image functors of all maps in $D^{\leq 2}\text{Comm}_{\mathcal{C}}$ have right adjoints.

Let

$$\begin{array}{ccc} B & \xrightarrow{g'} & B' \\ \uparrow f & \searrow \varphi & \uparrow f' \\ A & \xrightarrow{g} & A' \end{array}$$

be a commutative square in $D^{\leq 2}\text{Comm}_{\mathcal{C}}$. Let $M \in D(B\text{-Mod})$. Then we have a base change morphism

$$g_* f^* M \rightarrow f'^* g'_* M$$

defined to be the adjoint of the natural map $f^* M \rightarrow f^* g'^* g'_* M \xrightarrow{\varphi} g^* f'^* g'_* M$ or equivalently of the map $f'_* g_* f^* M \xrightarrow{\varphi} g'_* f_* f^* M \rightarrow g'_* M$.

The base change morphism is natural with respect to composition of commutative squares.

The following statement is trivial in the context of usual commutative algebras, but is a rather strong structure result in our context.

Proposition 17. *Let the notation be as above. If the square is a homotopy pushout, then the base change morphism $g_*f^*M \rightarrow f'^*g'_*M$ is an isomorphism.*

The proof will be given in the next section.

Let $A \rightarrow B$ be a map in $D^{\leq 2}\text{Comm}_{\mathcal{C}}$. Let $M \in D(A\text{-Mod})$ and $N \in D(B\text{-Mod})$. There is a projection morphism

$$M \otimes_A f^*N \rightarrow f^*(f_*M \otimes_B N)$$

adjoint to the natural map $f_*(M \otimes_A f^*N) = f_*M \otimes_B f_*f^*N \rightarrow f_*M \otimes_B N$. Note that for B -modules M', N' there is a natural map $f^*M' \otimes_A f^*N' \rightarrow f^*(M' \otimes_B N')$, and the projection morphism is equivalently described as the composition $M \otimes_A f^*N \rightarrow f^*f_*M \otimes_A f^*N \rightarrow f^*(f_*M \otimes_B N)$.

Proposition 18. *Let the notation be as above. Then the projection morphism $M \otimes_A f^*N \rightarrow f^*(f_*M \otimes_B N)$ is an isomorphism.*

We give the proof in the next section.

Let a square in $D^{\leq 2}\text{Comm}_{\mathcal{C}}$ be given as above and let $M \in D(B\text{-Mod})$, $N \in D(A'\text{-Mod})$ and $P \in D(A\text{-Mod})$. Set $M' := f^*M$, $N' := g^*N$, $\tilde{M} := g'_*M$, $\tilde{N} := f'_*N$ and $\tilde{P} := g'_*f_*P \cong f'_*g_*P$.

Lemma 12. *Let the notation be as above. Then the diagram*

$$\begin{array}{ccccc} (M' \otimes_A P) \otimes_A N' & \longrightarrow & g^*(g_*(M' \otimes_A P) \otimes_{A'} N) & \longrightarrow & g^*f'^*((\tilde{M} \otimes_{B'} \tilde{P}) \otimes_{B'} \tilde{N}) \\ \uparrow & & & & \uparrow \\ M' \otimes_A (P \otimes_A N') & \longrightarrow & f^*(M \otimes_B f_*(P \otimes_A N')) & \longrightarrow & f^*g^*(\tilde{M} \otimes_{B'} (\tilde{P} \otimes_{B'} \tilde{N})) \end{array}$$

where in the first two horizontal maps the projection morphism is applied and in the second two an adjunction and the base change morphism, commutes.

Proof. Let $F := g^*f'^* \xrightarrow{\varphi} f^*g'^*$. One checks that both compositions are equal to the composition $M' \otimes_A P \otimes_A N' \rightarrow F^*\tilde{M} \otimes_A F^*\tilde{P} \otimes_A F^*\tilde{N} \rightarrow F^*(\tilde{M} \otimes_{B'} \tilde{P} \otimes_{B'} \tilde{N})$, where the first arrow is a tensor product of obvious objectwise morphisms. \square

Let $A \in D^{\leq 2}\text{Comm}_{\mathcal{C}}$. We can use the two Propositions above to give the natural functor $M : D\text{Comm}(A) \rightarrow D(A\text{-Mod})$ a symmetric monoidal structure with respect to the coproduct on $D\text{Comm}(A)$ and the tensor product \otimes_A on $D(A\text{-Mod})$: We use the fact that $D\text{Comm}(A)$ is equivalent to the 1-truncation of $A \downarrow D^{\leq 2}\text{Comm}_{\mathcal{C}}$. So let $B \leftarrow A \rightarrow C$ be a triangle in $D^{\leq 2}\text{Comm}_{\mathcal{C}}$ and complete it by a homotopy pushout to a square with upper right corner $B \sqcup_A C$. First we apply the base change isomorphism to the unit $\mathbb{1}_B$ in $D(B\text{-Mod})$, which says that there is a natural isomorphism

$$(C \rightarrow B \sqcup_A C)^*(\mathbb{1}_{B \sqcup_A C}) \cong (A \rightarrow C)_*(M(B)).$$

Applying $(A \rightarrow C)^*$ to the left hand side of this isomorphism we get $M(B \sqcup_A C)$, applying this map to the right hand side we get $M(B) \otimes_A M(C)$ by the projection formula. This establishes the isomorphism $M(B) \otimes_A M(C) \cong M(B \sqcup_A C)$. That this isomorphism respects the commutativity isomorphisms follows from Lemma 12 with $P = \mathbb{1}_A$. That it respects the associativity isomorphisms for objects $f : A \rightarrow B$, $h : A \rightarrow C$ and $g : A \rightarrow A'$ in $A \downarrow \text{Ho}^{\leq 2}\mathcal{D}$ also follows from Lemma 12 with $M = \mathbb{1}_B$, $N = \mathbb{1}_{A'}$ and $P = h^*\mathbb{1}_C$ and a diagram chase.

10. PROOFS

In this section we give the proofs of Propositions 17 and 18. Assume throughout that Assumption 1 is fulfilled.

We need the concept of operads in $A\text{-Mod}$ for $A \in \text{Comm}_{\mathcal{C}}$. We also give the definition of a pointed operad, because it is needed in the Appendix. In the context of symmetric monoidal categories with pseudo-unit a pointed operad is not just an operad \mathcal{O} together with a pointing of $\mathcal{O}(0)$, the domains of the structure maps also have to be adjusted (see below).

So let us fix $A \in \text{Comm}_{\mathcal{C}}$. Let $A\text{-Mod}^u$ be the category of pointed A -modules, i.e. the category of objects in $A\text{-Mod}$ under A . For M a pointed or unpointed A -module and N a pointed or unpointed A -module let $M \otimes N$ be either $M \boxtimes_A N$, $M \triangleleft_A N$, $M \triangleright_A N$ or $M \boxdot_A N$, depending on whether M and N are unpointed, M is pointed and N is unpointed, M is unpointed and N is pointed or M and N are pointed. $M \otimes N$ is an object in $A\text{-Mod}$ unless both M and N are pointed in which case it is an object in $A\text{-Mod}^u$. Note that for M_1, \dots, M_n A -modules each of them either pointed or unpointed the product $M_1 \otimes \dots \otimes M_n$ is well defined, despite the fact that for different bracketings of this expression the symbols for which \otimes actually stands can be different.

Definition 12. An operad \mathcal{O} in $A\text{-Mod}$ is an object $\mathcal{O}(n) \in (A\text{-Mod})[\Sigma_n]$ for each $n \in \mathbb{N}$, where $\mathcal{O}(1)$ is pointed, together with maps

$$\mathcal{O}(m) \otimes \mathcal{O}(n_1) \otimes \dots \otimes \mathcal{O}(n_m) \rightarrow \mathcal{O}(n),$$

where $m, n_1, \dots, n_m \in \mathbb{N}$ and $n = \sum_{i=1}^m n_i$, such that the usual diagrams for these structure maps commute. A pointed operad in $A\text{-Mod}$ is the same as above with the exception that $\mathcal{O}(0)$ is also pointed.

Let $\text{Op}(A\text{-Mod})$ be the category of operads in $A\text{-Mod}$ and $\text{Op}^p(A\text{-Mod})$ the category of pointed operads in $A\text{-Mod}$. A pointed operad \mathcal{O} in $A\text{-Mod}$ is called *unital* if the pointing $A \rightarrow \mathcal{O}(0)$ is an isomorphism. Let $\text{Op}^u(A\text{-Mod})$ be the category of unital operads in $A\text{-Mod}$.

Let $(A\text{-Mod})^{\Sigma, \bullet\bullet}$ be the category of collections of objects $\mathcal{O}(n) \in (A\text{-Mod})[\Sigma_n]$, which are pointed for $n = 0, 1$ and unpointed otherwise. As for ordinary operads we have free (pointed) operad functors F starting from the categories $(A\text{-Mod})^{\mathbb{N}}$, $(A\text{-Mod})^{\Sigma}$, $(A\text{-Mod})^{\Sigma, \bullet\bullet}$ in the pointed case and various other pointed versions of these categories to $\text{Op}(A\text{-Mod})$ or $\text{Op}^p(A\text{-Mod})$. Note that if A is cofibrant all these source categories of the functors F are model categories.

Theorem 7. Let A be cofibrant in $\text{Comm}_{\mathcal{C}}$. Then the category $\text{Op}(A\text{-Mod})$ (resp. $\text{Op}^p(A\text{-Mod})$) is a cofibrantly generated J -semi model category over $(A\text{-Mod})^{\Sigma, \bullet}$ (resp. over $(A\text{-Mod})^{\Sigma, \bullet\bullet}$) with generating cofibrations $FF_A I$ and generating trivial cofibrations $FF_A J$. If \mathcal{C} is left proper, then $\text{Op}(A\text{-Mod})$ (resp. $\text{Op}^p(A\text{-Mod})$) is left proper relative to $(A\text{-Mod})^{\Sigma, \bullet\bullet}$ (resp. relative to $(A\text{-Mod})^{\Sigma, \bullet}$). If \mathcal{C} is right proper, so are $\text{Op}(A\text{-Mod})$ and $\text{Op}^p(A\text{-Mod})$.

Let f be a map in $A\text{-Mod}$ or $A\text{-Mod}^u$ and let g be a map in $A\text{-Mod}$ or $A\text{-Mod}^u$. Let $f \square_* g$ be the pushout product of f and g with respect to the product \otimes . $f \square_* g$ is a map in $A\text{-Mod}$ unless both f and g are maps in $A\text{-Mod}^u$ in which case $f \square_* g$ is a map in $A\text{-Mod}^u$.

Note that if A is cofibrant the category $A\text{-Mod}^u$ has a natural model structure as category of objects under A in the model category $A\text{-Mod}$. Note however that $A\text{-Mod}^u$ is *not* symmetric monoidal (with potential tensor product \boxtimes_A), since this product is not closed.

Lemma 13. *Let A be cofibrant in $\text{Comm}_{\mathcal{C}}$, let f be a cofibration in $A\text{-Mod}$ or $A\text{-Mod}^u$, let g be a cofibration in $A\text{-Mod}$ or $A\text{-Mod}^u$ let M be cofibrant in $A\text{-Mod}$ or $A\text{-Mod}^u$ and let N be cofibrant in $A\text{-Mod}$ or $A\text{-Mod}^u$. Then*

- *the pushout product $f\boxtimes_*g$ is a cofibration in $A\text{-Mod}$ or $A\text{-Mod}^u$ which is trivial if f or g is,*
- *the product $M\otimes f$ is a cofibration in $A\text{-Mod}$ or $A\text{-Mod}^u$ which is trivial if f is and*
- *the product $M\otimes N$ is cofibrant in $A\text{-Mod}$ or $A\text{-Mod}^u$.*

There is also a version of this statement when the map or object in $A\text{-Mod}$ has a right action of a discrete group G and the other map or object is in $A\text{-Mod}^u$ (resp. when both maps or objects are in $A\text{-Mod}$ and have actions of discrete groups G and G'). The resulting map or object is then a cofibration or cofibrant object in $(A\text{-Mod})[G]$ (resp. $(A\text{-Mod})[G \times G']$).

Note that in a symmetric monoidal category cases 2 and 3 would be special cases of case 1.

Proof. It suffices to show this for relative cell complexes f and g and cell complexes M and N , for which it follows for the first case by writing the pushout product of a λ -sequence and a μ -sequence as a $\lambda \times \mu$ -sequence. Let $M \in A\text{-Mod}^u$. Then if $A \rightarrow M$ is a λ -sequence, M itself is a $(1 + \lambda)$ -sequence in $A\text{-Mod}$. One concludes now by writing the products in cases 2 and 3 again as appropriate sequences. The cases with group actions work in the same way. \square

We remark now that there are versions of Propositions 5 and 6 for $\text{Op}^p(A\text{-Mod})$ where all tensor products are replaced by \otimes -products and all pushout products by the \otimes -pushout product \boxtimes_* . There is also a version of Lemma 3, from which Theorem 7 follows in the same way as Theorem 3.

Definition 13. *Let $\mathcal{O} \in \text{Op}(A\text{-Mod})$ (resp. $\mathcal{O} \in \text{Op}^p(A\text{-Mod})$).*

1. *An \mathcal{O} -algebra is an object $B \in A\text{-Mod}$ (resp. $B \in A\text{-Mod}^u$) together with maps*

$$\mathcal{O}(n) \otimes B^{\otimes n} \rightarrow A$$

satisfying the usual identities. The category of \mathcal{O} -algebras is denoted by $\text{Alg}(\mathcal{O})$.

2. *Let $B \in \text{Alg}(\mathcal{O})$. A B -module is an object $M \in A\text{-Mod}$ together with maps*

$$\mathcal{O}(n+1) \otimes B^{\otimes n} \otimes M \rightarrow M$$

satisfying the usual identities. The category of B -modules is denoted by $B\text{-Mod}$.

Let $\mathcal{O} \in \text{Op}^{(p)}(A\text{-Mod})$. The free \mathcal{O} -algebra functor $F_{\mathcal{O}} : A\text{-Mod} \rightarrow \text{Alg}(\mathcal{O})$ is given by

$$F_{\mathcal{O}}(M) = \coprod_{n \geq 0} \mathcal{O}(n) \otimes_{\Sigma_n} M^{\boxtimes n}.$$

In the pointed case $F_{\mathcal{O}}$ factors through $A\text{-Mod}^u$.

As in section 4 one shows the

Theorem 8. *Let A be cofibrant in $\text{Comm}_{\mathcal{C}}$ and let $\mathcal{O} \in \text{Op}(A\text{-Mod})$ (resp. $\mathcal{O} \in \text{Op}^p(A\text{-Mod})$).*

1. *If \mathcal{O} is cofibrant the category $\text{Alg}(\mathcal{O})$ is a cofibrantly generated J -semi model category over $A\text{-Mod}$ with generating cofibrations $F_{\mathcal{O}}F_AI$ and generating trivial cofibrations $F_{\mathcal{O}}F_AJ$. If \mathcal{C} is left proper (resp. right proper), then $\text{Alg}(\mathcal{O})$ is left proper relative to $A\text{-Mod}$ (resp. right proper).*
2. *Let \mathcal{O} be cofibrant as an object in $(A\text{-Mod})^{\Sigma, \bullet}$ (resp. in $(A\text{-Mod})^{\Sigma, \bullet\bullet}$). Then $\text{Alg}(\mathcal{O})$ is a cofibrantly generated J -semi model category with generating cofibrations $F_{\mathcal{O}}F_AI$ and generating trivial cofibrations $F_{\mathcal{O}}F_AJ$. If \mathcal{C} is right proper, so is $\text{Alg}(\mathcal{O})$.*

Let $\mathcal{N}_A \in \text{Op}(A\text{-Mod})$ (resp. $\mathcal{N}_A^u \in \text{Op}^u(A\text{-Mod})$) be the operad with $\mathcal{N}_A(n) = A$ (resp. $\mathcal{N}_A^u(n) = A$) for $n \in \mathbb{N}$ and the natural structure maps. Note that both categories $\text{Alg}^{(u)}(\mathcal{N}_A)$ are *not* equivalent to the category $\text{Comm}(A)$ but there are functors

$$C_{\mathcal{N}_A}^{(u)} : \text{Alg}(\mathcal{N}_A^{(u)}) \rightarrow \text{Comm}(A),$$

which are defined to be the left adjoints of the pullback functors $\text{Comm}(A) \rightarrow \text{Alg}(\mathcal{N}_A^{(u)})$. These adjoints exist since they exist on free algebras and every algebra is a coequalizer of two maps between free algebras (as is always the case for algebras over a monad).

Let $\mathcal{O} \in \text{Op}^{(p)}(A\text{-Mod})$ and $B \in \text{Alg}(\mathcal{O})$. As for ordinary algebras one defines the universal enveloping algebra $U_{\mathcal{O}}(B)$ as the quotient of the tensor algebra

$$\coprod_{n \geq 0} \mathcal{O}(n+1) \otimes_{\Sigma_n} B^{\otimes n}$$

by the usual relations. $U_{\mathcal{O}}(B)$ is an associative unital algebra in $A\text{-Mod}$, hence it is an A_{∞} -algebra in \mathcal{C} (i.e. an algebra over the operad \mathcal{L} considered as a *non- Σ* operad), which also has a universal enveloping algebra $U_{\mathcal{L}}(U_{\mathcal{O}}(B)) \in \text{Ass}(\mathcal{C})$. One has canonical equivalences

$$B\text{-Mod} \sim U_{\mathcal{O}}(B)\text{-Mod} \sim U_{\mathcal{L}}(U_{\mathcal{O}}(B))\text{-Mod}.$$

Let $F_B : A\text{-Mod} \rightarrow B\text{-Mod}$ be the free B -module functor.

As in section 6 one shows the

Theorem 9. *Let A be cofibrant in $\text{Comm}_{\mathcal{C}}$, let $\mathcal{O} \in \text{Op}(A\text{-Mod})$ (resp. $\mathcal{O} \in \text{Op}^p(A\text{-Mod})$) and $B \in \text{Alg}(\mathcal{O})$. Let one of the following two conditions be satisfied:*

1. *\mathcal{O} is cofibrant as an object in $(A\text{-Mod})^{\Sigma, \bullet}$ (resp. in $(A\text{-Mod})^{\Sigma, \bullet\bullet}$) and B is a cofibrant \mathcal{O} -algebra.*
2. *\mathcal{O} is cofibrant in $\text{Op}(A\text{-Mod})$ (resp. $\text{Op}^p(A\text{-Mod})$) and A is cofibrant as an object in $A\text{-Mod}$ (resp. in $A\text{-Mod}^u$).*

Then there is cofibrantly generated model structure on $B\text{-Mod}$ with generating cofibrations $F_B F_AI$ and generating trivial cofibrations $F_B F_AJ$.

Definition 14. An E_∞ -operad (resp. pointed E_∞ -operad) in $A\text{-Mod}$ is an object $\mathcal{O} \in \text{Op}(A\text{-Mod})$ (resp. $\mathcal{O} \in \text{Op}^p(A\text{-Mod})$) which is cofibrant as an object in $(A\text{-Mod})^{\Sigma, \bullet}$ (resp. in $(A\text{-Mod})^{\Sigma, \bullet\bullet}$) together with a map $\mathcal{O} \rightarrow \mathcal{N}_A$ which is a weak equivalence. A pointed E_∞ -operad \mathcal{O} is called unital if it is unital as an object in $\text{Op}^p(A\text{-Mod})$.

For \mathcal{O} a pointed E_∞ -operad in $A\text{-Mod}$ let us define the operad $\tilde{\mathcal{O}}$ in the same way as in section 8. Then we have analogues of Lemmas 7 and 8 and Corollary 5. So we are able to construct a unital E_∞ -operad in $A\text{-Mod}$ by first taking a cofibrant resolution $\mathcal{O} \rightarrow \mathcal{N}_A$ in $\text{Op}(A\text{-Mod})$ and then forming $\tilde{\mathcal{O}}$. This will be relevant in the Appendix.

Let $B \in \text{Alg}(\mathcal{O})$ be cofibrant. As in Lemma 10 one can show that the map $U_{\mathcal{O}}(B) \rightarrow B$ adjoint to the pointing $A \rightarrow B$ is a weak equivalence.

For the rest of this section let us fix an unpointed E_∞ -operad \mathcal{O} in $A\text{-Mod}$ (we could also take a pointed one). Let π be the map $\mathcal{O} \rightarrow \mathcal{N}_A$.

Lemma 14. Let A be cofibrant in $\text{Comm}_{\mathcal{C}}$. Then the composition

$$\text{Alg}(\mathcal{O}) \xrightarrow{\pi_*} \text{Alg}(\mathcal{N}_A) \xrightarrow{C_{\mathcal{N}_A}} \text{Comm}(A)$$

is a Quillen equivalence.

Proof. This follows from the fact that for a cofibrant A -module M the map

$$\mathcal{O}(n) \otimes_{\Sigma_n} M^{\boxtimes_{A^n}} \rightarrow M^{\boxtimes_{A^n}} / \Sigma_n$$

is a weak equivalence. \square

Lemma 15. Let A be cofibrant in $\text{Comm}_{\mathcal{C}}$ and let $B \in \text{Alg}(\mathcal{O})$ be cofibrant. Then the functor

$$B\text{-Mod} \rightarrow (C_{\mathcal{N}_A} \circ \pi)_*(B)\text{-Mod}$$

is a Quillen equivalence.

Proof. This follows from the fact that the map $U_{\mathcal{L}}(U_{\mathcal{O}}(B)) \rightarrow U_{\mathcal{L}}((C_{\mathcal{N}_A} \circ \pi)_*(B))$ is a weak equivalence, which follows itself from the description of these algebras in terms of transfinite compositions as in Propositions 8 and 12. \square

Lemma 16. Let A be cofibrant in $\text{Comm}_{\mathcal{C}}$ and let $B \in \text{Alg}(\mathcal{O})$ be cofibrant. Then $U_{\mathcal{O}}(B)$ is cofibrant as object in $A\text{-Mod}^u$.

Proof. Follows by the description of $U_{\mathcal{O}}(B)$ as in Proposition 12. \square

Corollary 9. Let A be cofibrant in $\text{Comm}_{\mathcal{C}}$. Then for cofibrant $B \in \text{Alg}(\mathcal{O})$ and cofibrant $M \in B\text{-Mod}$ the underlying A -module M is cofibrant in $A\text{-Mod}$.

Proof. Follows from Lemmas 16 and 13 and transfinite induction. \square

Lemma 17. Let μ and λ be ordinals and let $S_{\mu,+}$ and $S_{\lambda,+}$ be as in Proposition 8. Then there is a (necessarily unique) isomorphism

$$\varphi : S_{\lambda+\mu,+} \cong S_{\mu,+} \times S_{\lambda,+}$$

of well-ordered sets.

Proof. There is a natural inclusion $S_{\lambda,+} \hookrightarrow S_{\lambda+\mu,+}$, and φ maps its image to $\{*\} \times S_{\lambda,+}$ in the natural way. Now let $f \in S_{\lambda+\mu}$ with $f(i) \notin \mathbb{N}$ for some $i \in \mu$. There is a segment $M_f \subset S_{\lambda+\mu}$ starting at f which is isomorphic to $S_{\lambda,+}$ as a well-ordered set. Via this identification S_{λ} corresponds to all $f' \in S_{\lambda+\mu}$ with $f'|_{\mu} = f|_{\mu} + \frac{1}{2}$. Then φ maps M_f to $\{f|_{\mu}\} \times S_{\lambda,+}$ if $i > 0$ and to $\{f|_{\mu} - \frac{1}{2}\} \times S_{\lambda,+}$ if $i = 0$. It is easy to see that this way φ is well-defined, bijective and order-preserving. \square

Remark 6. If $f \in S_{\lambda,+}$ and $g \in S_{\mu,+}$ are successors, then φ maps $(f \sqcup g) - 1$ to $(g - 1, f - 1)$.

Proof of Proposition 17. By Lemmas 14 and 15 we can work in $\text{Alg}(\mathcal{O})$. So let $B, C \in \text{Alg}(\mathcal{O})$ be cofibrant. Let us denote the coproduct in $\text{Alg}(\mathcal{O})$ by \sqcup_A . We have to prove the base change isomorphism for the diagram

$$\begin{array}{ccc} B & \xrightarrow{g'} & B \sqcup_A C \\ \uparrow f & & \uparrow f' \\ A & \xrightarrow{g} & C \end{array} .$$

Let $M \in B\text{-Mod}$ be cofibrant. Then f^*M is cofibrant in $A\text{-Mod}$ by Corollary 9. Hence the base change morphism is represented by the morphism of $U_{\mathcal{O}}(C)$ -modules $U_{\mathcal{O}}(C) \triangleleft_A M \rightarrow U_{\mathcal{O}}(B \sqcup_A C) \triangleleft_{U_{\mathcal{O}}(B)} M$ which is adjoint to the map $M \cong A \triangleleft_A M \rightarrow U_{\mathcal{O}}(B \sqcup_A C) \triangleleft_{U_{\mathcal{O}}(B)} M$. We can assume that M is a cell module. Then by transfinite induction we are reduced to the following statement: Let $K \in A\text{-Mod}$ be cofibrant. Then the map $U_{\mathcal{O}}(C) \triangleleft_A (U_{\mathcal{O}}(B) \triangleleft_A K) \rightarrow U_{\mathcal{O}}(B \sqcup_A C) \triangleleft_A K$ is a weak equivalence. By Lemma 13 this follows if we show that the map of B -modules $\psi : U_{\mathcal{O}}(B) \boxtimes_A U_{\mathcal{O}}(C) \rightarrow U_{\mathcal{O}}(B \sqcup_A C)$ (where we exchanged the roles of B and C) is a weak equivalence. It suffices to prove this for cell algebras B and C . So let $B = \text{colim}_{i < \lambda} B_i$, where the transition maps are given by pushouts by cofibrations $g_i : K_i \rightarrow L_i$ in $A\text{-Mod}$ with cofibrant domain as in Proposition 8. Similarly let $C = \text{colim}_{i < \mu} C_i$, where the transition maps are given by pushouts by cofibrations $h_i : M_i \rightarrow N_i$ in $A\text{-Mod}$ with cofibrant domain. Then the map $0 \rightarrow U_{\mathcal{O}}(B \sqcup_A C)$ is described as in Proposition 12 by a $S_{\lambda+\mu,+}$ -sequence (1). Since the maps $0 \rightarrow U_{\mathcal{O}}(B)$ resp. $0 \rightarrow U_{\mathcal{O}}(C)$ are $S_{\lambda,+}$ - resp. $S_{\mu,+}$ -sequences, the map $0 \rightarrow U_{\mathcal{O}}(B) \boxtimes_A U_{\mathcal{O}}(C)$ is a $S_{\mu,+} \times S_{\lambda,+}$ -sequence (2) by Lemma 1 (this also holds in the case of a symmetric monoidal category with pseudo-unit). Let $\alpha : S_{\mu,+} \times S_{\lambda,+} \rightarrow S_{\lambda+\mu,+}$ be the isomorphism of well-ordered sets of Lemma 17. Let $f \in S_{\lambda,+}$ and $f' \in S_{\mu,+}$ be successors. Then α identifies $(f \sqcup f') - 1$ and $(f' - 1, f - 1)$, and the relevant pushouts in the sequences (1) and (2) are by maps

$$\mathcal{O}(|f \sqcup f'| + 1) \otimes_{\Sigma_{f \sqcup f'}} \square_{*i < \lambda} g_i^{\square_* f(i)} \square_* \square_{*i < \mu} h_i^{\square_* f'(i)} \quad \text{and}$$

$$\mathcal{O}(|f| + 1) \otimes \mathcal{O}(|f'| + 1) \otimes_{\Sigma_f \times \Sigma_{f'}} \square_{*i < \lambda} g_i^{\square_* f(i)} \square_* \square_{*i < \mu} h_i^{\square_* f'(i)} .$$

It is easy to see by transfinite induction that the map ψ is compatible with sequences (1) and (2) via the identification α on the indexing sets and with the above pushouts by the map induced by the tensor multiplication map $\mathcal{O}(|f| + 1) \otimes \mathcal{O}(|f'| + 1) \rightarrow \mathcal{O}(|f \sqcup f'| + 1)$ which inserts the second object into the last slot of the first object. This map is a weak equivalence because \mathcal{O} is an E_{∞} -operad, hence the claim follows by transfinite induction. \square

Proof of Proposition 18. By Lemmas 14 and 15 we can assume that we have a cofibrant $\tilde{B} \in \text{Alg}(\mathcal{O})$, a cofibrant $\tilde{N} \in \tilde{B}\text{-Mod}$ and a cofibrant $M \in A\text{-Mod}$ and prove the projection isomorphism for M and the image N of \tilde{N} in $B\text{-Mod}$, where B is the image of \tilde{B} in $\text{Comm}(A)$. Since \tilde{N} is cofibrant as A -module by Corollary 9 the projection morphism is represented by the composition

$$M \boxtimes_A \tilde{N} \rightarrow M \boxtimes_A N \cong (B \triangleleft_A M) \boxtimes_B N,$$

where the isomorphism at the second place is from Lemma 11. So we have to show that the first map is a weak equivalence. We can assume that \tilde{N} is a cell module. Then by transfinite induction one is left to show that for a cofibrant A -module K the map $M \boxtimes_A (U_{\mathcal{O}}(\tilde{B}) \triangleleft_A K) \rightarrow M \boxtimes_A (B \triangleleft_A K)$ is a weak equivalence. But this map is the map from the free \tilde{B} -module on $M \boxtimes_A K$ to the free B -module on $M \boxtimes_A K$, which is a weak equivalence by Lemma 15. Hence we are finished. \square

11. APPENDIX

Assume that Assumption 1 is fulfilled.

In this section we give an alternative definition of a product on the derived category of modules over an algebra in $D^{\leq 2}\text{Comm}_{\mathcal{C}} := D^{\leq 2}\text{Alg}(\mathcal{N})$ without using the special properties of the linear isometries operad. Unfortunately it seems to be rather ugly (or difficult) to construct associativity and commutativity isomorphisms, and we did not try hard to do this! Note that $D^{\leq 2}\text{Comm}_{\mathcal{C}}$ is the same up to canonical equivalence as the category denoted with the same symbol in section 9. If \mathcal{O} is a unital E_{∞} -operad and $A \in D^{\leq 2}\text{Comm}_{\mathcal{C}}$, then there is a representative $\tilde{A} \in \text{Ho}^{\leq 2}\text{Alg}(\mathcal{O})$ which is well defined up to an isomorphism which itself is well defined up to a unique 2-isomorphism. There is a similar statement for a lift of A into $\text{Alg}(\mathcal{O})$.

Let us first treat the case where \mathcal{C} is simplicial, since it is a bit nicer. Let \mathcal{O} be a pointed E_{∞} -operad in \mathbf{SSet} and denote by \mathcal{O} also its image in $\text{Op}(\mathcal{C})$. In \mathbf{SSet} the diagonal $\Delta : \mathcal{O} \rightarrow \mathcal{O} \times \mathcal{O}$ is a map of operads, hence we also have a map of operads $\mathcal{O} \rightarrow \mathcal{O} \otimes \mathcal{O}$ in $\text{Op}(\mathcal{C})$.

We will define a tensor product on $\text{Ho } A\text{-Mod}$ for a cofibrant \mathcal{O} -algebra A .

First note that for \mathcal{O} -algebras A and B the tensor product $A \otimes B$ is a $\mathcal{O} \otimes \mathcal{O}$ -algebra, hence also a \mathcal{O} -algebra via Δ . Also for an A -module M and a B -module N the tensor product $M \otimes N$ has a natural structure of an $A \otimes B$ -module. If A, B are unital there are induced maps in $\text{Alg}^u(\mathcal{O})$ $A = A \otimes \mathbb{1} \rightarrow A \otimes B$ and $B = \mathbb{1} \otimes B \rightarrow A \otimes B$.

Proposition 19. *Assume that \mathcal{O} is either unital or cofibrant in $\text{Op}(\mathcal{C})$. Let $A, B \in \text{Alg}^u(\mathcal{O})$ be cofibrant. Then the canonical map $A \sqcup B \rightarrow A \otimes B$ in $\text{Alg}^u(\mathcal{O})$ induced by the maps $A \rightarrow A \otimes B$ and $B \rightarrow A \otimes B$ is a weak equivalence.*

Proof. This proof is very similar to a part of the proof of Proposition 17. By Lemma 7 we are reduced to the case where \mathcal{O} is unital. It suffices to prove the claim for cell algebras A and B . So let $A = \text{colim}_{i < \lambda} A_i$, where the transition maps are given by pushouts by maps $g_i : K_i \rightarrow L_i$ as in Proposition 8. Similarly let $B = \text{colim}_{i < \mu} B_i$, where the transition maps are given by pushouts by maps $h_i : M_i \rightarrow N_i$. Then the map $0 \rightarrow A \sqcup B$ is described by Proposition 8 by a $S_{\lambda+\mu,+}$ -sequence (1). Since the

maps $0 \rightarrow A$ resp. $0 \rightarrow B$ are $S_{\lambda,+}$ - resp. $S_{\mu,+}$ -sequences, the map $0 \rightarrow A \otimes B$ is a $S_{\mu,+} \times S_{\lambda,+}$ -sequence (2). Let $\alpha : S_{\lambda+\mu,+} \rightarrow S_{\lambda,+} \times S_{\mu,+}$ be the isomorphism of well-ordered sets of Lemma 17. Let $f \in S_{\lambda,+}$ and $f' \in S_{\mu,+}$ be successors. Then α identifies $(f \sqcup f') - 1$ and $(f' - 1, f - 1)$. The relevant pushouts in the sequences (1) and (2) are by maps

$$\begin{aligned} & \mathcal{O}(|f \sqcup f'|) \otimes_{\Sigma_{f \sqcup f'}} \square_{i < \lambda} g_i^{\square f(i)} \square \square_{i < \mu} h_i^{\square f'(i)} \quad \text{and} \\ & \mathcal{O}(|f|) \otimes \mathcal{O}(|f'|) \otimes_{\Sigma_f \times \Sigma_{f'}} \square_{i < \lambda} g_i^{\square f(i)} \square \square_{i < \mu} h_i^{\square f'(i)}, \end{aligned}$$

and again one shows by transfinite induction that the map $\psi : A \sqcup B \rightarrow A \otimes B$ is compatible with sequences (1) and (2) via the identification α on the indexing sets and with the above pushouts by the map induced by

$$\mathcal{O}(|f| + |f'|) \xrightarrow{\Delta} \mathcal{O}(|f| + |f'|) \otimes \mathcal{O}(|f| + |f'|) \xrightarrow{\beta \otimes \gamma} \mathcal{O}(|f|) \otimes \mathcal{O}(|f'|),$$

where β inserts the pointing $\mathbb{1} \rightarrow \mathcal{O}(0)$ into the last $|f'|$ slots of $\mathcal{O}(|f| + |f'|)$ and γ inserts the pointing into the first $|f|$ slots. This map is a weak equivalence since \mathcal{O} is an E_∞ -operad, so our claim follows by transfinite induction and the assumptions. \square

Assume that \mathcal{O} is either unital or cofibrant in $\text{Op}(\mathcal{C})$. For any cofibrant \mathcal{O} -algebra A let Q_A denote a cofibrant replacement functor in $A\text{-Mod}$. Let $A \in \text{Alg}^u(\mathcal{O})$ be cofibrant. Then the map $A \sqcup A \rightarrow A \otimes A$ is a weak equivalence. Now define a functor

$$\begin{aligned} T : A\text{-Mod} \times A\text{-Mod} &\rightarrow A\text{-Mod} \quad \text{by} \\ T(M, N) &:= (A \sqcup A \rightarrow A)_*(Q_{(A \sqcup A)}(Q_A M \otimes Q_A N)). \end{aligned}$$

It is clear that T descends to a functor

$$T : D(A\text{-Mod}) \times D(A\text{-Mod}) \rightarrow D(A\text{-Mod}).$$

We will see that this functor is naturally isomorphic to the tensor product defined in section 9.

Now we skip the restriction of \mathcal{C} being simplicial. Let \mathcal{O} be a unital E_∞ -operad in \mathcal{C} which always exists by Lemma 7. Then the operad $\mathcal{O} \otimes \mathcal{O}$ is also a unital E_∞ -operad. Let $A, B \in \text{Alg}(\mathcal{O} \otimes \mathcal{O})$. Let $\pi_1 : \mathcal{O} \otimes \mathcal{O} \rightarrow \mathcal{O} \otimes \mathcal{N} \cong \mathcal{O}$ and $\pi_2 : \mathcal{O} \otimes \mathcal{O} \rightarrow \mathcal{N} \otimes \mathcal{O} \cong \mathcal{O}$ be the two projections and define $A_i := \pi_{i,*} A$, $B_i := \pi_{i,*} B$, $i = 1, 2$. Note that π_1 and π_2 are weak equivalences. There are maps

$$\begin{aligned} A_1 \otimes \mathbb{1} &\rightarrow A_1 \otimes B_2 \quad \text{and} \\ \mathbb{1} \otimes B_2 &\rightarrow A_1 \otimes B_2 \end{aligned}$$

of $\mathcal{O} \otimes \mathcal{O}$ -algebras and natural isomorphisms of $\mathcal{O} \otimes \mathcal{O}$ -algebras $A_1 \otimes \mathbb{1} \cong \pi_1^* A_1$ and $\mathbb{1} \otimes B_2 \cong \pi_2^* B_2$, which are on the underlying objects in \mathcal{C} the isomorphisms $A_1 \otimes \mathbb{1} \cong A_1$ and $\mathbb{1} \otimes B_2 \cong B_2$. Using the adjunction units $A \rightarrow \pi_1^* A_1$ and $B \rightarrow \pi_2^* B_2$ we finally get maps $A \rightarrow A_1 \otimes B_2$ and $B \rightarrow A_1 \otimes B_2$, hence a map

$$A \sqcup B \rightarrow A_1 \otimes B_2$$

of $\mathcal{O} \otimes \mathcal{O}$ -algebras.

Proposition 20. *Let $A, B \in \text{Alg}(\mathcal{O} \otimes \mathcal{O})$ be cofibrant. Then the map $A \sqcup B \rightarrow A_1 \otimes B_2$ constructed above is a weak equivalence.*

Proof. The proof of this Proposition is exactly the same as the one for Proposition 19, except that this time the relevant pushouts in the sequences (1) and (2) are by maps

$$(\mathcal{O}(|f| \sqcup |f'|) \otimes \mathcal{O}(|f| \sqcup |f'|)) \otimes_{\Sigma_{f \sqcup f'}} \square_{i < \lambda} g_i^{\square f(i)} \square \square_{i < \mu} h_i^{\square f'(i)} \quad \text{and} \\ \mathcal{O}(|f|) \otimes \mathcal{O}(|f'|) \otimes_{\Sigma_f \times \Sigma_{f'}} \square_{i < \lambda} g_i^{\square f(i)} \square \square_{i < \mu} h_i^{\square f'(i)} .$$

The map $A \sqcup B \rightarrow A_1 \otimes B_2$ is again compatible with these pushouts by the map induced by

$$\mathcal{O}(|f| + |f'|) \otimes \mathcal{O}(|f| + |f'|) \xrightarrow{\beta \otimes \gamma} \mathcal{O}(|f|) \otimes \mathcal{O}(|f'|) ,$$

where β inserts the pointing $\mathbb{1} \rightarrow \mathcal{O}(0)$ into the last $|f'|$ slots of $\mathcal{O}(|f| + |f'|)$ and γ inserts the pointing into the first $|f|$ slots. This map is again a weak equivalence since \mathcal{O} is an E_∞ -operad, so we are done. \square

Let $D\text{Comm}_{\mathcal{C}} := D\text{Alg}(\mathcal{N})$.

Corollary 10. *The natural functor $M : D\text{Comm}_{\mathcal{C}} \rightarrow \text{Ho } \mathcal{C}$ has a natural symmetric monoidal structure with respect to the coproduct on $D\text{Comm}_{\mathcal{C}}$ and the tensor product on $\text{Ho } \mathcal{C}$.*

If \mathbb{S} -modules are available in \mathcal{C} it is clear that this symmetric monoidal structure is naturally isomorphic to the one constructed at the end of section 9.

Let now $A \in \text{Alg}(\mathcal{O} \otimes \mathcal{O})$ be cofibrant. Note that for $M, N \in A\text{-Mod}$ the tensor product $\pi_{1,*}M \otimes \pi_{2,*}N$ is an $A_1 \otimes A_2$ -module, hence also an $A \sqcup A$ -module. Consider the functor

$$T : A\text{-Mod} \times A\text{-Mod} \rightarrow A\text{-Mod} , \\ (M, N) \mapsto (A \sqcup A \rightarrow A)_*(Q_{A \sqcup A}(\pi_{1,*}(Q_A M) \otimes \pi_{2,*}(Q_A N))) .$$

It is again clear that T descends to a functor

$$T : D(A\text{-Mod}) \times D(A\text{-Mod}) \rightarrow D(A\text{-Mod}) .$$

To see that this functor is isomorphic to the previous functor T in the simplicial case one takes the previous \mathcal{O} to be $\mathcal{O} \otimes \mathcal{O}$ and looks at the map of $\mathcal{O} \otimes \mathcal{O}$ -algebras (obtained via the diagonal) $A \otimes A \rightarrow (A_1 \otimes \mathbb{1}) \otimes (\mathbb{1} \otimes A_2)$. The last algebra is isomorphic to the $\mathcal{O} \otimes \mathcal{O}$ -algebra $A_1 \otimes A_2$. Hence for A -modules M and N we get a map of $A \otimes A$ -modules $M \otimes N \rightarrow M_1 \otimes N_2$ which is a weak equivalence. From this one gets the natural isomorphism we wanted to construct.

It remains to show that in the cases \mathcal{C} receives a symmetric monoidal left Quillen functor from \mathbf{SSet} or $\text{Comp}_{\geq 0}(\mathbf{Ab})$ the functor T is isomorphic to the tensor product \otimes_A defined in section 9.

To do this let \mathcal{O} be a unital E_∞ -operad in $\mathbb{S}\text{-Mod} = \mathbb{1}_{\mathbb{S}}\text{-Mod}$ and let $\overline{\mathcal{O}} := \mathcal{O} \otimes_{\mathbb{S}} \mathbb{1}$ be its image in $\text{Op}(\mathcal{C})$. The operad $\mathcal{O} \otimes \mathcal{O}$ (which is defined componentwise) is also a unital E_∞ -operad whose image in $\text{Op}(\mathcal{C})$ is $\overline{\mathcal{O}} \otimes \overline{\mathcal{O}}$. Then by the above procedure one can define a tensor product on $\text{Ho}(A\text{-Mod})$ for a cofibrant $\mathcal{O} \otimes \mathcal{O}$ -algebra A , and it is easy to see that this coincides (after the appropriate identifications) with the product T defined above on $\text{Ho}(\overline{A}\text{-Mod})$ (\overline{A} is the image of A in $\text{Alg}(\overline{\mathcal{O}} \otimes \overline{\mathcal{O}})$) on the one hand and with the product $\boxtimes_{A'}$ on $\text{Ho}(A'\text{-Mod})$, where A' is the image of A in $\text{Comm}_{\mathcal{C}}$, on the other hand.

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